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An Examination of Asset Mispricing in a Simple One-Asset Market

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An Examination of Asset Mispricing in a Simple One-Asset Market

Abstract
A framework is developed that allows the use of various valuation methods and pricing schemes. The framework is then applied to two simple one-asset models. These models are analyzed to see how changing valuations and the existence of cognitive biases such as an endowment effect and an availability heuristic can affect future prices. Instead of searching for an equilibrium and proving its stability this paper examines what causes deviations from equilibrium. Additionally, a stochastic differential equation is developed to model how group populations change over time, such as noise trader populations, and to introduce evolution into a simple model. Graphs of the stochastic differential equation are presented and are used to examine whether arbitrageurs can eliminate noise traders from the market. The key features of this thesis are the application of ideas in new ways and the discussion of current and suggested methods in the literature.

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LAKE FOREST COLLEGE

Senior Thesis

An Examination of Asset Mispricing in a Simple One-Asset Market

by

Daniel Acevski

April 20, 2015

The report of the investigation undertaken as a Senior Thesis, to carry one course of credit in the Department of Economics

_________________________________________  _____________________________________________
Michael T. Orr  Jeffrey Sundberg, Chairperson
Krebs Provost and Dean of the Faculty

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Kent Grote
Abstract

A framework is developed that allows the use of various valuation methods and pricing schemes. The framework is then applied to two simple one-asset models. These models are analyzed to see how changing valuations and the existence of cognitive biases such as an endowment effect and an availability heuristic can affect future prices. Instead of searching for an equilibrium and proving its stability this paper examines what causes deviations from equilibrium. Additionally, a stochastic differential equation is developed to model how group populations change over time, such as noise trader populations, and to introduce evolution into a simple model. Graphs of the stochastic differential equation are presented and are used to examine whether arbitrageurs can eliminate noise traders from the market. The key features of this thesis are the application of ideas in new ways and the discussion of current and suggested methods in the literature.
I wish to dedicate this thesis to my family for being wonderful people and supporting me in this endeavor. I couldn’t have done this alone.

Love truly can make miracles happen.
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I wish to thank my friends as well. They provided me encouragement at times when it was much needed.
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Introduction

The reason for this composition is because I believe that recent literature regarding market efficiency and asset pricing is not moving in the correct direction. There is a shift towards including evolution in models, as advocated by (Lo, 2004), and behavioral models have indeed become much more popular. However, the majority of the literature makes the same assumptions and takes very similar approaches. Additionally, the focus in the efficient markets literature has been mostly empirical. I am not of the mind that the prevailing approaches to modeling markets should be abandoned, but I believe that people should take up some new approaches. The most appealing approaches are those that use evolution, such as Andrew Lo’s Adaptive Market Hypothesis (Lo, 2004). Most evolutionary models are numerical instead of analytical, which is not a problem; evolutionary models are difficult to solve. However, this should not discourage people from attempting to make analytically tractable, evolutionary models. Overall, the models presented here are, for the most part, not analytically tractable. A stochastic differential equation is used in this paper to model the evolution of a group population over time and to create an evolutionary model. Stochastic differential equations are prevalent in finance but they are used to model price movements, not population changes. I believe that using them to model population changes will be an important tool in the future. Graphical analysis, also used in this paper, is also a tool I have not seen in the literature, and using it with supply and demand yielded acceptable results. By attempting to figure out how the market supply and demand change with regard to certain factors it is possible to examine shifts from equilibrium. Much of the current literature has not focused on such shifts from equilibrium, but has instead focused on establishing the existence of equilibria under certain assumptions, and showing that said equilibria are stable. If the equilibria are stable then deviations are not a concern, but what if the assumptions do not hold, as time has proven of many models. I believe that there needs to be changes made regarding the approaches used in the literature and, while this paper does not provide a complex or analytically tractable model, a framework is developed that yields insight into new approaches and ideas.
Review of Literature

The idea that markets fully reflect all available information has been around for many years now. This idea, commonly referred to as the Efficient Market Hypothesis, is strongly attributed to Eugene F. Fama. Fama’s contribution to the study of efficient markets is indeed great – it earned him a Nobel Prize in Economic Sciences in 2013 – and he has written some very comprehensive papers on the subject such as (Fama, 1970) and (Fama, 1991). His work, along with Paul Samuelson’s "Proof that properly anticipated prices fluctuate randomly" (Samuelson, 1965) forms the foundation of the efficient market hypothesis. Indeed, many aspects from these original papers still remain today, such as Fama’s characterization of efficiency into weak-form – the information set contains only historical prices – semi-strong from – where any publicly available information is considered – and strong-form, which considers whether private information is incorporated into prices.

Fama and Samuelson may have introduced the Efficient Market Hypothesis, but many other researchers have made significant contributions to the idea. For example, (LeRoy, 1973), (Rubinstein, 1976), and (Lucas, 1978) extend Fama’s and Samuelson’s frameworks to allow for risk-averse investors. The idea here is that many people dislike risk and require compensation for taking risk. Indeed, now including risk aversion in a model is standard, and is often done via a utility function. Another paper from this time period that included risk aversion is "On the Impossibility of Informationally Efficient Markets," (Grossman & Stiglitz, 1980). I bring up this paper not only because it is well written, but also because the title illustrates an issue regarding efficient markets: many people believe that markets are not efficient. Indeed many contributions to the Efficient Markets Hypothesis have come from challenges to the theory and rebuttals. These challenges arose from the difficulty of testing whether markets are efficient and also from the developments in economics over the past 30 to 40 years.

The reason that empirically testing the efficiency of a market is difficult is due to the joint-hypothesis problem discussed in (Fama, 1991). Basically, when one conducts an empirical test of the efficient market hypothesis they are conducting a test of the model used
to describe the market as well. If the empirical results disagree with the Efficient Market Hypothesis the reason could be that the model used is incorrect, not that markets are inefficient. (Summers, 1986) discusses testing market efficiency and deals with this issue by assuming that ex ante returns are constant and known. (Lo, 2008) nicely articulates the problem caused by the Efficient Market Hypothesis being a joint-hypothesis: any test of market efficiency "must concern the kind of information reflected in prices, and how this information comes to be reflected in prices." (Konté, 2010), a much more recent paper, examines market efficiency independently of the joint-hypothesis using stochastic differential equations to show that the joint hypothesis may not be the reason why many empirical studies find markets to be inefficient. He also suggests using an evolutionary approach to modeling market efficiency.

(Konté, 2010) illustrates two important facts: there are different ways to model efficient markets, and stochastic differential equations may be useful tools. Regarding the different ways to model efficient markets, some papers assume that all agents are rational, which is the approach used in early papers. Other publications search for a rational expectations equilibrium (Palmer, Arthur, Holland, LeBaron, and Tayler, 1994) or a Bayesian Nash equilibrium. Some publications simply assume Bayesian learning (Blume and Easley, 1992; see also Bloomfield, Libby, and Nelson, 2000; Ludwig and Zimper, 2013; Shefrin and Statman, 1994), or instead assume heterogeneous beliefs (Chiarella and He, 2002; Hirshleifer and Luo, 2001; Kyle, 1985). Lastly, some papers attempt evolutionary models (Konté, 2010; see also Lo, 2004).

The study of non-rational beliefs has arisen from the recent growth of behavioral economics and the application of psychology to economics (Kahneman and Tversky, 1979; Thaler, 2005). As far as I know there is much less literature on using evolutionary models to study market efficiency; I can only think of (Konté, 2010), (Lo, 2004), and (Hommes, 2013), as well as some newer articles by Lo which mainly re-iterate why evolutionary models should be used. I have seen mention of evolutionary models in older papers, but usually analytical solutions are not examined. Similarly, there is a lack of presence of stochastic
differential equations in the literature of efficient markets,¹ but there are cases of them being used and referenced in the past, and they seem to be becoming more popular, as evidenced by (Konté, 2010).

Before moving on to the next section I want to discuss two more aspects of the literature. The first is the idea of noise trading: noise trading is basically the idea that not all people trade on information or, if they do, some people trade on bad information. The idea of noise trading was first introduced in (Black, 1986). Since then it has been focused on quite a bit in the efficient markets literature since, if noise traders exist, they will likely make markets inefficient. Many people believe, however, that noise traders will be forced out of the market because they will lose money. (Shleifer, 2000) provides a good introduction to both noise trading and inefficient markets. The second aspect I want to mention is that there are not a large variety of models of efficient markets. There is indeed a wide body of literature, but most papers present empirical analyses of market efficiency, often through examining how a specific event influences the price of an asset – what Fama termed "event studies" – and the papers that do present analytical work often present modifications of already established models, changing certain assumptions such as heterogeneous beliefs or rational expectations. There does not seem to be much work in the examination or creation of new models, and not many models seem to incorporate the number of noise traders, or how the population of noise traders evolves. (Shleifer, 2000) contains one such model, but I do not know of another example. There are quite a few studies, however, that present models allowing the existence of noise traders. (Luo, 2012) presents such a model. Andrew Lo, for example in (Lo, 2004), often mentions that the literature should shift to evolutionary models and argues why this is the case. However, as stated earlier there are few analytical overviews of evolutionary models – a fact mentioned by Lo himself²

¹Although, in general, they are used quite a lot in financial economics
²Lo says that there is little qualitative study of such evolutionary models. There are quantitative studies, especially given the emergence of programs such a Mathematica
General Framework

Consider the market for a single asset and let there be $N_t$ participants in this market at time $t$. Each participant, $i$, has a total wealth at time $t$ of $W_i^t$, and he invests a fraction of this wealth, $y_i^t$, in the asset at time $t$. The decision of what fraction of wealth to invest in the asset is exogenous; it is not determined in the model.

The reason I specify a participant’s total wealth and the fraction of it he invests instead of simply saying that participant $i$ invests $s$ dollars in the asset is because I wish to create a general framework – a framework that can accommodate a variety of models. Specifying the total amount of wealth, while unnecessary at this point, will more easily allow the application of the framework to models with wealth-maximizing behavior during future work. Keeping the framework general will also aid in the future extension of the model to a more complex, multi-asset market. Once such an extension is made, the fractional wealth term $y_i^t$ will allow examination of how the fraction of wealth invested in various assets changes over time. An extension of the model to a more complex market would be useful since applying the model to every asset in a larger and more complex market would yield an indication of mispricing across the entire market. For example, one could examine how much each asset is mispriced and use this information to determine the average mispricing across the market. The issue with such a generalization, however, is that a single asset in a market is not isolated from all other assets, so the current one-asset framework may not apply. More specifically, the presence of another asset could affect how much it and other assets are mispriced. To see this consider two very similar stocks, $\alpha$ and $\beta$ in the same sector: if $\beta$ is performing average while $\alpha$ is performing exceptionally well, investors may expect $\beta$ to perform well in the future as well. The expectation that $\beta$ will perform well in the future could cause people to overvalue $\beta$ and lead to mispricing. Such an effect cannot occur in a one-asset model. I lessen this problem by including in the model a variable describing the strength of whatever market sector the single asset is in. This lessons the problem because the market sector performing well is similar to $\alpha$ performing well while $\beta$ is performing average – if $\beta$ is performing average while the market sector is performing
well then people should expect β’s performance to improve.

Let time be discrete and indexed by \( t, t = 1, 2, \ldots \). Time is assumed to be discrete because we can make the interval from \( t-1 \) to \( t \) very small and thus approximate continuous time. More specifically, in simulations I consider the interval from \( t-1 \) to \( t \) to be .001 seconds. I also assume that there is no predefined time horizon: in principle the market continues forever. I consider such a short time interval because I am interested in day trading. With day trading, even if individual investors are not acting so quickly, with a large number of traders in the market it is possible for new orders to be placed or cancelled within such short intervals. Indeed, as the number of participants in a market goes to infinity it should be expected that the frequency of new orders would increase. The only time where this would not be the case would be if a restriction that all traders submit orders during the same time period is placed in the model. A time interval of .001 seconds is also appropriate for modeling the presence of both long term traders and day-traders since the day traders will be acting quickly. For modeling only long-term traders – that is, traders who hold the asset for long periods of time – one could make the interval from \( t-1 \) to \( t \) be on the order of days or weeks.

The participants in the considered one asset market consist of only buyers and sellers. Given our earlier notation of \( N_t \) being the total number of participants, let \( m_t \) be the total number of sellers, and \( n_t \) be the total number of buyers, all in period \( t \), \( N_t, m_t, n_t \in \mathbb{Z}^+ \). Therefore, \( N_t = m_t + n_t, t = 1, 2, \ldots \). Furthermore, at the beginning of the first period \((t = 1)\), all market participants have information regarding some existing price of the asset, denoted \( p_0 \). I note this simply to establish that there exists some past information regarding the stock: we are not discussing an asset that does not have any history. This assumption is important because later in the paper, as the model is established, investor’s valuations of the asset are important. Accordingly, there will be discussion regarding how the asset is valued. Each individual’s valuation is largely determined by factors that depend on the asset already having been introduced to the market, such as the past history of prices or knowledge of dividends. If I do not assume that the asset has a history at the start of my model, then I would not be able to accurately model the first period, time \( t = 1 \). An inaccurate history
would thus be created – period 1 would be incorrect, which would affect period 2 and make
it incorrect, and so on ad infinitum – and the model’s accuracy would be greatly affected.
It is because of this that I assume the asset has some history. This is not a poor assumption,
since the only time an asset would not have some a history would be if it was a new listing.
If we require that the asset have a certain amount of history before modeling it, then we are
simply restricting the model to assets of a certain age. The amount of time an asset needs
to be on the market to accumulate a price history, news, and expectations is fairly short, so
this is not an issue.

At each time \( t \), each participant – whether a buyer or a seller – holds a number of shares
of the asset, denoted by \( h_i(t), i \in N \ h_i(t) \geq 0 \). This is \( \geq \) because a buyer may or may
not possess any holdings, but a seller must have some. Furthermore, if \( s q^i_t \) is the number
of shares listed – being sold – by participant \( i \) at time \( t \), then \( 0 \leq s q^i_t \leq h_i(t) \). These two
conditions ensure that short selling does not occur in our model.

In the model a participant decides to buy or sell a stock based on whether he expects a
greater wealth from doing so rather than not doing so. In other words, participants act so as
either increase their wealth or prevent it from decreasing. Buyers are the ones who act to
increase their wealth – they will buy if they think buying the stock will yield higher future
wealth – and sellers act to prevent decreases of wealth – if they expect the asset to be worth
less in the future they will sell. When an individual, \( i \), holds both money and the asset, his
wealth can be specified using the value of the asset. If \( t \) is the current period then the value
of the asset is taken to be the current price of the asset \( p_t \). Investor \( i \)'s wealth at time \( t \) can
then be written as

\[
W^i_t = M^i_t + h_i(t)p_t
\]

where \( M^i_t \) is how much money investor \( i \) has at time \( t \), \( h_i(t) \) is how many shares of the asset
he has at time \( t \). If \( t \) is the current period and we are interested in what individual \( i \) expects
his wealth to be at time \( t + 1 \), we use a term that represents individual \( i \)'s belief of what the
value of the asset will be at time \( t + 1 \), denoted \( v^i_{t+1} \). This belief is formed at time \( t \). Using
this variable we can write investor $i$’s expectation of his wealth at time $t$ as

$$W_{t+1}^i = M_{t+1}^i + h_i(t + 1)v_{t+1}^i$$

(a)

I call this an expectation of investor $i$’s future wealth because his belief of the value of the asset in the next period, $v_{t+1}^i$ could be incorrect. Similarly, we could write investor $i$’s expected future wealth from the point of view of a different investor, $j$, $i \neq j$. This would be $W_{t+1}^i = M_{t+1}^i + h_i(t + 1)v_{t+1}^j$. We could also define the actual value of the asset in the next period, $v_{t+1}^*$, in which case investor $i$’s actual wealth in the next period is $W_{t+1}^i = M_{t+1}^i + h_i(t + 1)v_{t+1}^*$. It is currently assumed that there is no time value of money; if investor $i$ does not buy or sell any shares of stock between time $t$ and $t + 1$ then $M_{t+1}^i = M_{t+1}^i$. Under this assumption any increase in wealth will occur when the stock price increases and any decrease in wealth will occur when the stock price decreases. Since no interest is earned on money an investor will maximize his wealth by holding only the asset when the price will increase and holding only cash when the price will decrease. Since investor $i$ wants to maximize his wealth, he will buy when he expects the price to increase and sell when he expects the price to decrease.

I wish to point out that each individual’s belief of the future value of the asset, $v_{t+1}^i$, is allowed to differ because I believe this is a source of mispricing – the phenomena I wish to model. Granted, there can be mispricing even if all investors agree on the future value of the asset; the agreed upon price would simply have to be incorrect. It would be a strange phenomenon for all investors to agree on a value of the asset, however, and even stranger for every single investor to incorrectly predict the future price. Consequently, I am not interested in this case. It is much more likely for mispricing to arise from some investors valuing the asset correctly and other investors valuing the asset incorrectly.

The other assumptions that I have made so far are also justifiable. Assuming that the current value of the asset is the current price, $p_t$, is permissible because the possibility of dividends and risk having been taken into account during the formulation of $p_t$ is not precluded. The assumption that the current value of the asset equals the current price is
essentially an assumption that the current price is efficient. Neither the future price nor the market is necessarily efficient, however. A third assumption made is that an investor determines his belief of what the asset will be worth at time \( t + 1 \), \( v_{t+1}^i \), before the price at time \( t + 1 \) is be revealed. I make this assumption because, in the models below, the valuation of all individuals is what determines the price at time \( t + 1 \). It is demand and supply that drive prices, not prices that drive demand and supply.

One aspect to note is that, in the model that follows, the price of the asset in period \( t + 1 \) will depend on how market participants value the asset; the future price will depend on \( v_{t+1}^i \), \( \forall i \in N \). Depending on how specific one wants to be, the model can assume that investors do not consider this, that they do consider this, or that they consider this and also model how they consider it. I examine the case where investors do not take this into account. This assumption is believable because it is difficult for real investors to predict their effect on the market. For example, using the model in this paper they would need to know each investor’s valuation of the asset and each investor’s wealth. Nobody can posses this information, except for, in my model at least, the market maker.

To conclude this section I will explain the order of events in the models that will follow. Let \( t \) be the current period. An assumption that was stated earlier said that in period \( t \) there is already information about the asset. Denote the set of this information by \( \theta \). At the end of time \( t \) – that is, right before time \( t + 1 \) begins – each market participant formulates a prediction regarding the value of the asset will be at time \( t + 1 \). Each market participant uses their valuation – their prediction of the what the value will be in the future – to estimate their future wealth and therefore to decide whether they will bid for shares or ask to sell shares and what fraction of wealth they invest. The investors then place limit orders accordingly.\(^3\) The orders they place will be executed up to (down to) individual \( i \)'s valuation if he is a buyer (seller). Time \( t + 1 \) still hasn’t begun when these orders are placed. The market maker then uses the information gained from the order placed to determine \( p_{t+1} \). Time \( t + 1 \) then arrives and price \( p_{t+1} \) is set. Shares are then bought and sold at the beginning of period.

\(^3\) Limit orders are the only order type allowed in the models presented because they are consistent with the idea of extrapolating demand and supply curves presented later.
$t + 1$ based on the orders placed. Naturally, they are bought and sold at price $p_{t+1}$, the price at time $t + 1$. There is no requirement that all orders are filled; supply does not have to meet demand. This requirement can be implemented when applying the framework to a specific model, if such behavior is desired. How the price at time $t + 1$, $p_{t+1}$, is set, and how individuals determine the value of the asset at time $t + 1$ will also be specified when the framework is applied to specific models.

**Types of Participants**

As mentioned in the previous section there are buyers and sellers. We assume that no participant is both a buyer and a seller in the same period. If a participant wanted to both buy and sell shares during the same period he could simply buy/sell fewer shares (since the buying and selling would be happening at the same price). We model this assumption by basing investor decisions to buy or sell on their valuation of the asset: an investor buys the asset if they believe it will increase in value and sell otherwise. Since it is impossible to expect an asset to both increase and decrease in value no investor will want to both buy and sell shares in the same period, $t$. In reality, however, investors may engage in buying and selling behavior to hedge against risk. In a stock market an investor could short-sell a stock to finance the purchase of another. In an option market an investor may buy and sell options with slightly different expiration dates or strike prices. Such behavior is not possible in a one-asset model. Naturally, this is a limitation of the current model.

While the model currently only has two trader types, buyers and sellers, later in the paper we decompose buyers and sellers into smaller groups based on how they create a valuation of the asset. Stating this in our established notation, we group investors $i, j \in N$, $i \neq j$, based on how they determine the future value of the asset, $v_{t+1}^i$ and $v_{t+1}^j$. If $v_{t+1}^i = v_{t+1}^j$ then investors $i, j$ get grouped together. Groups can be as small as 1 investor or as large as $N$ investors.

In reality it is very unlikely for one person’s valuation to not coincide with any other person’s valuation. This is because the market has many participants, and as the number
of participants increases the likelihood of two people having the same valuation should increase. If there were one person in the market with a different valuation from everybody else, however, it would likely be because of noise. This noise could have various sources, one of which is cognitive biases such as the availability heuristic and the anchoring affect. I consider such biases sources of noise because, according to Black, "noise is what makes our observations imperfect" (Black, 1986). The availability heuristic could cause one individual to have a different valuation from everybody else because, in the extreme case, perhaps a single individual lived through events that nobody else in the market has. This could be The Great Depression if one person in the market is much older than everyone else. The anchoring affect could cause an individual’s valuation to be unique because people perhaps anchor to different things.

A group having \( N_t \) investors – all investors being in the same group – is also not realistic. This event would likely occur only if there is no noise and every investor used the same valuation technique. For every investor to use one technique, however, that technique would need to have been proven overwhelmingly superior to every other technique. In fact, that technique would need to produce 100% accurate predictions regarding the future value of the asset. The technique being 100% accurate means that no individual would have an incentive to develop another technique. If the technique was slightly less accurate, say only 99% accurate, then while individuals would not have a large incentive to develop a different technique, an incentive would still exist and some investors would eventually deviate from the accepted valuation – the 99% accurate one – in expectation of it being wrong and them making profits. If we allow there to be noise in the market, then even if a technique is 100% accurate valuations could deviate because of noise. Worth noting, however, is that even with noise everyone’s valuations can coincide – everybody can have zero noise for a particular period, or every investor’s noise could coincide (systematic noise). Both of these possibilities are unlikely, but I point them out because they are not often stated explicitly in the literature. This is perhaps because they are trivial considerations, but if a market is typically noisy and, if for some strange reason, all noise coincides for a period an analyst could be led to believe that the market is less noisy than it truly is.
Another problem with the case of all investors having the same valuation is that there would not be a market. Our model predicts that in such a case there would either be demand and no supply or supply and no demand. This is also a realistic prediction because stocks are not a necessity; nobody will buy stocks if they expect the value to decrease and nobody will sell stocks if they expect the value to increase. If all people have the same beliefs then there will not be both demand and supply.

Forming Valuations

Earlier we said that \( v_{t+1}^i \) is what participant \( i \) believes the asset will be worth at time \( t + 1 \). How is \( v_{t+1}^i \) determined? Previously, I assumed that the value of the asset at time \( t \) is \( p_t \). Similarly, I assume that the value of the asset at time \( t + 1 \) is \( p_{t+1} \), so price and value are interchangeable terms in this paper. Asking how each individual estimates what the value of the asset will be at time \( t + 1 \) is therefore the same as asking what they believe the price will be. How investor \( i \) forms a belief of the future price depends on the model to which the framework is being applied. For now it is assumed that

\[
v_{t+1}^i = p_t + \gamma_i^t
\]

where \( p_t \) is the price at time \( t \) and \( \gamma_i^t \) is a term characterizing investor \( i \)'s belief about how much the value of the asset will change from period \( t \) to \( t + 1 \), \( \gamma_i^t = v_{t+1} - p_t, \gamma \in \mathbb{R} \). In words, investor \( i \) estimates the future price based on the current price and a belief of how much the price will change. In this price formation, the current price serves as an "anchor" in that people use it to determine the future price. The logic supporting this is that people should be much more likely to accept a prediction close to the current price than a prediction far from the current price. For example, if a stock is trading at $50 and someone comes up with a prediction of $5 for the future price, they will doubt their prediction. Only if the investor believes that the stock is greatly mispriced or has the utmost faith in his method of prediction will he not suspect a mistake in his prediction. Such strong faith in one’s prediction is not precluded by the model; \( \gamma_i^t \) can be sufficiently large to represent
such a case. The introduction of anchoring and the assumption of such a simple rule is meant to model the fact that all investors, even professional investors, likely partake in satisficing.\(^4\) Note that \(\gamma^t_i \geq -p_t\) as nobody will predict a negative price and, while a prediction of 0 is highly unlikely, it is possible for someone to expect the issuer of the stock to go out of business or become unlisted. Furthermore, the definition of \(v^t_{i+1}\) can be changed to suit different situations. For example, one can define \(v^t_{i+1}\) based on risk, where \(v^t_{i+1}\) increases or decreases as risk increases. An investor with such a \(v^t_{i+1}\) would be risk seeking or averse. Another possibility would be to specify that investor \(i\)'s valuation, \(v^t_{i+1}\) is completely independent of risk, and that investor \(i\) dislikes risk. Such a case may lead to a model similar to that of mean-variance maximizing, a fact that is illustrated by the indifference curves for these assumptions, presented in Figure 1.

![Figure 1: Indifference Curves When Risk and Valuation Are Separate](image)

Although the way in which investors predict the future price is simple, their determination of \(\gamma_t\) can be as complicated as a model requires. In the model considered below it is assumed that \(\gamma^t_i = f(\Gamma^t_i, \mu^t_i, \theta_t)\), where \(\Gamma^t_i\) represents past experiences of participant \(i\), \(\mu^t_i\) represents an endowment effect from participant \(i\)'s holdings \(h_i(t)\), and \(\theta_t\) represents

\(^4\)For more on satisficing see (Simon, 1956).
all other information available at time \( t \), such as earnings announcements and the current price. The fact that all other information affects price is trivial, The fact that the current price affects investor \( i \)’s prediction of the future price besides by providing an anchoring effect is not trivial. In addition to providing a benchmark for investor \( i \)’s future price estimate via an anchoring affect, the current price can also influence how much investor \( i \) believes the price will change. A simple example that illustrates this would be if investor \( i \)’s beliefs exhibited some sort of simple, positive bias. For example, assume that investor \( i \) believes that the price will increase by 1% every period. If \( p_t = 100 \), then investor \( i \) will predict a $1 price change, but if \( p_t = 10 \) then investor \( i \) will predict a 10 cent price change. This example illustrates how an investor’s belief regarding the degree of a future price change can depend on the current price. Another non-trivial characteristic regarding the inclusion of \( \theta_t \) is modeling the effect all other information has on price. The cause of this is the ambiguity of all other information. In order to handle this difficulty I recommend specifying what "other information" is allowed in a model. Over time, as the framework is developed, more types of "other information" can be incorporated into models. In the simulations referenced in this paper I will also assume either that there is no outside information, or that the only outside information is the current and past prices.

In addition to other information such as price history, earnings announcements, and dividend announcements, investor \( i \)’s belief regarding how much the price will change in the next period is influenced by \( \Gamma \) and \( \mu \). \( \Gamma \) and \( \mu \) are both effects that are included out of consideration for behavioral economics.\(^5\) \( \Gamma \) accounts for an availability heuristic. When \( \Gamma_t \) refers to a negative experience, such as remembering a recession, then investor \( i \) is more likely to underestimate a price increase or overestimate a price decrease; he will be pessimistic regarding the market. The event(s) that \( \Gamma \) refers to need not be negative, however. Perhaps a recent boom or bull market is fresh in an investor’s memory, or is all that the investor has experienced. In such a case investor \( i \) is likely to overestimate a price increase or underestimate a price decrease. \( \mu \) accounts for a possible endowment effect. I am not aware of any studies on endowment effects in asset markets, but they should exist.

\(^5\)For a refresher on behavioral economics see (Thaler, 2005) and (Kahneman & Tversky, 1979).
For example, if a person owns a stock he should feel that the stock is worth more than it is; he will not want to admit that he bought a stock that is not a "winner". Even if the stock has gone up in price and is a "winner", an investor may think that the price will increase further. He may think that there is something special or great about the stock he bought. This effect may also be proportional to the number of shares investor $i$ owns of the asset, $h_i(t)$ or the total wealth he has invested in the asset, $y^i_t W^i_t$. The more an investor has at stake the larger his endowment effect may be. The endowment effect could be crucial if an objective prediction of the future asset price indicates a slight decrease. If investor $i$ believes that the asset is worth more than it truly is because of an endowment effect, and if his overestimate of the future value of the asset because of the endowment effect is greater than the predicted price decrease, then investor $i$ may fail to realize that the price will decrease and thus hold on to an asset that he should sell. An example of this situation would be if there is a $1 endowment effect on a $20 asset – the investor feels the asset is worth $21 once he buys it because of an endowment effect. A 50 cent price decrease is then predicted, so the asset is predicted to be worth $19.50. The investor may feel that the asset will be worth $20.50 in the future and thus holds onto the asset instead of choosing to sell it.

Recall that we asserted $\gamma^i_t = f(\Gamma^i_t, \mu^i_t, \vartheta_t)$. In light of this assertion equation 1 can be rewritten as

$$v^i_{t+1} = p_t + f(\Gamma^i_t, \mu^i_t, \vartheta_t)$$

(1*)

Note that this assertion implies that the distribution of $\gamma^i_t$’s evolves over time (since past information and availability heuristics change over time) and that $\gamma^i_t$’s are not independent across time (because, as shall be seen later, trader assumptions of the asset value at time $t + 1$, $v^i_{t+1}$, affect price and thus get incorporated into $\gamma^i_t$). Therefore, the distribution of $\gamma^i_t$’s is neither independent nor identically distributed across time $t, t + 1, \ldots$. Also note that $\gamma^i_t$ is correlated with $p_t$ since $p_t$ affects how large $\gamma^i_t$ is.
Pricing

Earlier, it was established that an individual will buy the asset if he expects the price to increase and will sell if he expects the price to decrease. This is a direct result of our assumption that there is no time value of money, so holding shares when the price increases maximizes wealth and selling shares when the price decreases minimizes loss of wealth.

We also defined $v_{i}^{t+1}$ as individual i’s prediction of the value of the asset at time $t + 1$ made at time $t$. Note that $i \in \{1, 2, 3, ..., N_t\}$ where $N_t$ is the number of participants in the market at time $t$. These assumptions indicate that individual $i$ will be a buyer if $v_{i}^{t+1} > p_{t}$ and will be a seller if $v_{i+1} < p_{t}$. Given this information, Individual i’s demand and supply functions are assumed to be

$$d_{i}^{t}(p_{t}) = \begin{cases} \left[0, \frac{W_i^i}{v_{t+1}^{t+1}}\right] & \text{if } v_{i}^{t+1} > p_{t+1} \text{ and } p_{t} \\ 0 & \text{otherwise} \end{cases}$$

(2)

$$s_{j}^{t}(p_{t}) = \begin{cases} A_{j}^{i} \in \left[0, h_{i}(t)\right] & \text{if } v_{i}^{t+1} < p_{t+1} \text{ and } p_{t} \\ 0 & \text{otherwise} \end{cases}$$

(3)

where $d_{i}^{t}(p_{t})$ is buyer $i$’s demand for the asset at time $t$ given price $p_{t}$, the price at time $t$, and $s_{j}^{t}(p_{t})$ is the quantity of the asset supplied by seller $j$ at price $p_{t}$. $y_{i}$ is the fraction of wealth buyer $i$ chooses to invest in the asset at time $t$, and $W_i^i$ is buyer $i$’s total wealth at time $t$. The reason why the piecewise functions are defined in terms of both $p_{t+1}$ and $p_{t}$ is because we assumed that limit orders are placed, that transactions are made at $p_{t+1}$, and because a buyer will expect to make money as long as the price he pays is less than what he believes the future price will be while a seller will lose less money as long as he liquidates his holdings at a price above what he believes the asset will be worth in the future. This model does not allow investors to change their mind after placing an order and take the risk of trying to get a better price in the future if a price reversal occurs. Waiting is risky because instead of a price reversal the price could continue in the same direction. $A_{j}^{i}$ is
the number of shares seller \( j \) decides to sell in period \( t \) and must be less than the seller’s holdings, \( h_j(t) \). This restricts short selling.

The way that Equation 3 is written, the seller does not decide to sell different amounts depending on the realized price in time \( t + 1 \), \( p_{t+1} \): once the seller decides to offer some shares for sale in a period, he cannot alter his decision. Finally, I assume that the buyer cannot spend less than the amount of wealth he decided to invest in the asset at time \( t \), \( W^i_t = y^i_t \). Therefore, the buyer only faces the decision of when to buy, not how much to buy.

Under this assumption 2 can be rewritten as:

\[
d_q^i(p_t) = \begin{cases} 
\frac{y^i_t W^i_t}{v^i_{t+1}} & \text{if } V^i_t > p_{t+1} \text{ and } p_t \\
0 & \text{otherwise}
\end{cases}
\] (2*)

These are not new assumptions; they are implied by the earlier assumption that investors want to maximize wealth. If an investor expects a positive return on the asset he will maximize his next period wealth by converting all his cash into the asset, since there is no return on cash in the model. Similarly, if an investor expects a negative return he will minimize his losses by converting all his assets into cash. Therefore, individual \( i \)'s supply of the asset at time \( t \) can once more be re-written as

\[
s_q^i(p_t) = \begin{cases} 
h_i(t) & \text{if } V^i_{t+1} < p_{t+1} \text{ and } p_t \\
0 & \text{otherwise}
\end{cases}
\] (2*)

which says that an investor will list all his holdings for sale if he expects the value of the asset to be less in the next period than in the current period.

Our specified demand function will model investor behavior accurately if investors identify an asset in which they want to invest, determine how much money they wish to invest, and then wait for when they expect the value of the asset to increase before buying. This could be the case for investors who search for indications of an upcoming price increase; the fraction of wealth they invest in the asset can simply be made a function of
how strong the indicator is since an investor will want to invest more money if there is a
strong indication of a future price increase than if there is a weak indication. The supply
curve is not a great model of investor behavior, however. Investors will sell off all their
holdings if they expect a large price decrease, but if they are uncertain about the occurrence
or magnitude of a future price decrease they may choose to sell only part of their holdings.
This behavior is not considered by the current model, but can be incorporated by adding an
uncertainty term to the supply equation.

**Model 1: Pricing Rule Model**

Now that we have established individual demand and supply equations, our next goal is
to establish at what price the asset is traded. In actual markets the price at which a trade
occurs is set by a market maker.\(^6\) We will discuss this type of interaction later. For now, the
only participants in our market are buyers and sellers. In order to model the evolution of
the market price, and since we are constructing a simple model, we use a very simple price
adjustment scheme as in (Palmer et al., 1994)

\[
p_{t+1} = p_t (1 + \eta [B_t - O_t])
\]

(4)

Where \(B_t\) is the total quantity of bids and \(O_t\) is the total quantity of offers, both at time
\(t\). We chose to apply this price adjustment scheme to our model since we are looking for
both simplicity and a characterization of the evolution of prices. As Palmer points out, "\(\eta\)
is a crucial determinate of the ultimate behavior." Small \(\eta\) leads to very slow adjustment of
prices, while large \(\eta\) gives large oscillations."(Palmer et al., 1994). \(\eta\) can be modeled as an
adaptive mechanism, but here we keep it fixed. In order to model the evolution of prices
we need to find an expression for \(B_t - O_t\). The first step we take in finding an expression
for \(B_t - O_t\) is defining total supply and total demand. These are just the summation of

\(^6\)I will sometimes abbreviate market maker as M.M
individual supply and individual demand, respectively. Mathematically:

\[ Q_d^i(p_t) = \sum_N q_i^d(p_t) \] (5)

\[ Q_s^i(p_t) = \sum_N q_i^s(p_t) \] (6)

We can iterate the summation through all traders in the market, \( N \), because if an investor is a supplier in period \( t \) their demand will be 0, and if an investor is a buyer in period \( t \) their supply will be 0 – buyers in the supply equation and sellers in the demand equation do not affect aggregate demand and supply. Recalling that \( q_i^d(p_t) = \frac{y^i W_i}{\nu_{t+1}} \) and \( q_i^s(p_t) = h_i(t) \), we get that \( Q_d^i(p_t) = \sum_N \frac{y^i W_i}{\nu_{t+1}} \) and \( Q_s^i(p_t) = \sum_N h_i(t) \). Once we have this calculating \( B_t - O_t \) is simple; it is just

\[ \sum_N d_i^t(p_t) - \sum_N s_i^t(p_t) = \sum_N \frac{y^i W_i}{\nu_{t+1}} - \sum_N h_i(t) \] (4)

It is here that we first encounter the main problem with this model; it does not offer clear conclusions. What the above equation says is that the difference between the number of buy offers and sell offers is the difference between the summation of individual’s demand and the summation of individual’s supply. This is obvious. If we use a more specific model, however, we can get more significant conclusions. The beauty of the framework introduced above is that it can be easily used with more specific models.

To illustrate both that the framework can be easily used with more specific models and that using a more specific model yields more significant conclusions, consider the earlier definition that \( \nu_{t+1} = p_t + f(\Gamma_i^t, \mu_i^t, \theta_t) \), where \( \Gamma_i^t \) represents the past experiences of participant \( i \), \( \mu_i^t \) represents an endowment effect from participant \( i \)’s holdings \( h_i(t) \), and \( \theta_t \)

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7I use framework to refer to the overall technique and idea of using individual valuations, in general, to examine market effects. In the simple, one asset markets considered here, market effects are just the effect on the price and volume of the asset. I use the word model to refer to instances where the formations of valuations, prices, demand, and supply are specified.
represents all other information available at time $t$. Substituting this into (4) results in

$$\sum_{N} y_i^t W_i^t p_t + f(\Gamma_i^t, \mu_i^t, \vartheta_i) - \sum_{N} h_i(t)$$  \hspace{1cm} (7)$$

When viewing this equation, keep in mind that $\sum_{N} h_i(t)$ depends on $v_{t+1}$ as well – and thus on $p_{t+1}$. The above substitution allows us to see how the future price is affected when $p_t$ and $f(\cdot)$ change. If we explicitly specify $f(\cdot)$ then we can also see how the future price is affected by $\Gamma_i^t$, $\mu_i^t$, and $\vartheta_i$. With this in mind, we assume that $f(\cdot) = a \Gamma_i^t + b \mu_i^t + R(\vartheta_i)$, $\Gamma_i^t, R(\vartheta_i) \in \mathbb{R}$, $\mu_i^t \in [0, 1]$. $\mu_i^t$ must be positive because the more shares a person owns the higher they will value the asset, $\Gamma_i^t$, which represents the presence of an availability bias, can be negative or positive because an investor may have recently had either a positive or negative experience, and $R(\vartheta_i)$ represents a utilization of available information to get a better estimate of the future value of the asset.

$\mu_i^t$, always being positive, increases each individual’s valuation. If we make one more assumption that $y_i$ is an increasing function of $v_{t+1}$, the overall effect of $\mu_i^t$ will be to increase the future price. This is because, by increasing every investor’s valuation, it does not lower the quantity each buyer demands but it causes some would be sellers to become buyers. The effect will be especially large as the number of sellers whose valuations are slightly below $p_t$ increases, and when the difference between $p_t$ and $p_{t+1}$ is particularly small. There will be no effect if no seller’s valuation is close to $p_t$. The reason there will be a large effect when a seller’s valuation of the asset, $v_{t+1}$, is close to $p_t$ is that a small increase will cause him to predict a price increase and thus become a buyer. When a seller becomes a buyer $B_t - O_t$ increases and, by our pricing rule $p_{t+1} = p_t(1 + \eta[B_t - O_t])$.

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8This note being evident is the consequence of poor notation.

9A logical assumption since the more valuable an asset is the more people will want to buy. Without this assumption an increase in $v_{t+1}$ would lead to demanding less shares.

10Remember that an investor sells if he expects the value of the asset to decrease and buys if he expects the value of the asset to increase.
the future price increases. Figure 2 supports this effect, where the endowment effect is shown by the difference between the black and the blue lines.

To generate this data I specified an initial price of $50. For 1050 investors I then randomly generated holdings, \( h_i(t) \), between 1000 and 100,000, random real numbers between \( .00001 \) and \( .000025 \) for the endowment effect \( \mu \), an availability heuristic term between \( -.025 \) and \( .025 \), and wealth invested in the asset between 1,000 and 1,000,000 dollars. I specified the magnitude of the endowment effect as \( h_i(t) \) times the generated endowment effect term, the magnitude of the availability heuristic as the generated availability heuristic term times the current price, and \( p_t + R(\vartheta) \) by randomly generating values close to the current price – roughly within 5% of the current price. All random values were generated via Wolfram Mathematica’s RandomInteger and RandomReal functions. The reason why I generated valuations for \( p_t + R(\vartheta) \) is because I consider \( R(\vartheta) \) as some sort of "pricing rule" that investors use to help predict the future value of the asset. This is a logical assumption given that \( \vartheta \) represents all available information; investors do use all available information for forecasting the future price. I kept these forecasts within 5% of the current price, \( p_t \), because valuations should be pretty close to the current price. This is even truer since I wish to model day trading. In order to calculate each individual’s biased estimations I added the randomly generated \( p_t + R(\vartheta) \) to the calculated endowment and availability heuristic effect. That is, I assumed that 

\[
\nu_{t+1} = p_t + R(\vartheta) + h_i(t) \mu_i + \Gamma_i \cdot p_t.
\]

These terms are additive, not multiplicative, because an investor’s valuation should not go
to zero if one of these biases goes to zero, and none of the biases should have too large of an effect. Using $h_i(t)$ to weight $\mu_i$ makes sense since the endowment effect depends on a person's holdings, but using $p_t$ to weight $\Gamma'_i$ is a stretch because an investor may recall a time when a cheap stock’s price skyrocketed, and thus could predict a large price change for low-priced stocks. One flaw in the simulation is that, in order to compare the effect of biases, I kept $p_t + R(\vartheta)$ constant between simulations, $\forall t$. $R(\vartheta)$ is likely correlated with $p_t$, however, so keeping this sum constant increases the inaccuracy of the model. As I do not have a specific definition for $R(\vartheta)$, however, I could not make the sum variable.

Figure 2 also illustrates how investors using an availability heuristic can affect price, and that biases do affect price. Investors being subject to an availability heuristic, however, could either increase or decrease the future price. The direction of their impact would depend on whether the majority of investors are recalling positive events – and thus have a positive bias – or are recalling negative events. It also depends on how they are distributed between being buyers and sellers. This ambiguity is clear in Figure 2: when there is an endowment effect but no availability heuristic considered, the price is sometimes below the no bias price and sometimes above it. Additionally, the availability heuristic does not seem to have much of an effect in this model. Modeling the inclusion of both biases is interesting as the simulation shows that the biases can counteract each other. Look around time $t = 4$: the price when both biases are included is between the price when only one-bias is included. This implies that biases, and therefore noise, can indeed cancel each other out. It may also be possible for the biases to amplify each other. The reason for this is that the presence of a bias affects how wealth is distributed over time, and if wealth accumulates to more biased investors they will be able to have a larger effect. Neither the simulation nor the model say much about the likelihood of cancellation, however, and the effect of the biases may be exacerbated by the fact that this is a model where sellers must sell all of their holdings. Selling all of one’s holdings leads to accumulating larger amounts of wealth than normal.

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11The green line represents the price when there are no biases and the blue line represents the price when there is only the availability heuristic

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It is also possible to use the model to discuss market efficiency. A difficulty arises, however, when trying to evaluate the expected future price. Investors place bid and ask orders based on their expected future price, but the expected future price depends on the bid and ask orders they place. One unappealing solution would be to assume that all investors know the beliefs and preferences of all other investors. This is not a realistic assumption. Instead, I define an efficient price, $p_{t+1}^* = \mathbb{E}[p_{t+1} | \theta]$, where $\theta$ is the set of all available information, and use it as a benchmark. If $p_{t+1}$ is the efficient price, then we can assume that it is reached by investor $i$ valuing the asset at $v_{i,t+1}^*$, $\forall i$. If investors form this valuation without being biased, then introducing biases to the market will result in the effects discussed above; we will deviate from the efficient equilibrium. One investor producing a biased valuation typically will not affect the asset price significantly. This is not true, however, if the investor whose valuation is biased makes up a large portion of the buy or sell orders for the asset – the investor’s bias can change the asset price significant if he is a buyer and his wealth is much greater than the wealth of all other investors, or if he is a seller and owns many more shares than all other investors. If many investors are biased, however, then deviations will occur. The only way deviations would not occur is in the event that their biases cancel each other out. It is safe to say that, if the market is already at an efficient equilibrium, real people, with their biases, will cause the market to deviate from the efficient equilibrium.

**Model 2: Market Maker Model**

In this section we present a model where, instead of having the price follow a pricing rule, we have a third type of agent – a market maker – set the price. We already discussed biases and how they affect valuations in the previous model, so for this model we will simply discuss how changes in individuals’ valuations affect price. Furthermore, we discuss how noise and evolution can be incorporated into the model. We use a stochastic differential equation to introduce the idea of evolution – an application of stochastic differential equations to the modeling of noise that may be unique to this paper. First, however, we make two assumptions regarding the market maker to make analyzing the model easier:
i. The market maker sets price \( s.t \) the quantity traded is maximized

ii. The information the market maker receives regarding orders is the price at which each participant is willing to buy or sell the asset \((v_i^{t+1} \forall i)\), and the quantity they are willing to buy/sell at that price. He sees their limit orders.\(^{12}\)

Assumption (ii) implies that price is set such that total demand and total supply equal each other. In other words, the market maker sets a price corresponding to the intersection of supply and demand. This follows from the fact that the demand curve is monotonically decreasing.\(^{13}\)

\(^{12}\)The information seen by the market maker is shown in Figure 3.

\(^{13}\)See appendix A Theorem 1.0 for an overview of the proof.
Keeping the notation from earlier, we let $p_t$ be the current price of an asset at time $t$ and $N_t$ be the number of agents in the one-asset market at time $t$. Each agent’s belief of the price of the asset at time $t + 1$, formulated at time $t$, is denoted $v_{t+1}$. $v_{t+1}$ is formed similarly to how it was formed earlier in that each agent forms this belief, $v_{t+1}$, via the summation of the current price, $P_t$ and their belief regarding how much the price will change from time $t$ to time $t + 1$, denoted $b_i^t(x_{t-1},...,x_{t-L})$, as in (Hommes, 2013). That is, $v_{t+1} = p_t + b_i^t(x_{t-1},...,x_{t-L})$. However, we are not assuming the existence of biases such as an endowment effect; $b_i^t(x_{t-1},...,x_{t-L})$ will depend only on the trader type, $h$, of agent $i$, and is the same for every trader of type $h$. Therefore, $b_i^t(x_{t-1},...,x_{t-L})$ will be denoted as $b_t^h$ from now on. Time is considered discrete in this model, and $T = \mathbb{N}$ is used to denote the set of all possible times $t$. $t = 0$ is considered to be the first period, and we assume that the asset has a price $P_0$ at this time.

Every participant will be the same in the sense that they do not know the future price. Participants are differentiated by the fact that they formulate their belief of how much the price will change in the future, $b_t^h$, differently.\footnote{In the future is used to mean “at the start of the next period} This way of defining participants means that every agent is a noise trader. This idea is justified by the fact that nobody can know the future price exactly. Some people may predict the price change more accurately, however, which the model allows.\footnote{Arbitrage opportunities may exist, such as taking advantage of mergers, but we do not consider such cases explicitly in this paper} What this model will focus on is how a person switching from one trader type to another, or a trader types $b_t^h$ changing, affects price. From now on, superscript $i$ will be used to refer to agent $i \in N$. Subsets of $N$ may be defined and considered later in order to refer to groups of participants who formulate their beliefs in the same way. Additionally, the growth and decline of any group of people using the same rule (any subset of $N$) can be modeled using a stochastic differential equation. One obvious relationship is that the total amount of immigration/emigration in any time period is equal to the sum of immigration/emigration for every individual group. To illustrate this fact, assume that there are two groups in the market. Let $B \subset N$ be the set of people in the first group, and $C \subset N$ be the set of people in the second group, $B \cup C = N$. If 10 people leave
Demand and supply are assumed to follow the same functions defined earlier. That is, individual $i$’s demand and supply at time $t$ are, respectively, $\frac{y_i^t W_i^t}{v_i^{t+1}}$ if the current price is below his valuation, and $A_i^t = h_i(t)$ if the current price is above his valuation. Furthermore, we assume that there is a market maker who sets the price each period in order to maximize volume, which is the same as setting the price s.t. demand and supply intersect. Mathematically, this is represented as $\sum \frac{y_i^t W_i^t}{v_i^{t+1}} = \sum A_i^t$. In the event that the quantity of shares demanded is above (below) the quantity of shares supplied at every price then the market maker sets the price corresponding to the lowest (highest) bid price, as this will maximize the number of shares traded, given our assumption that demand is an increasing function and supply is a decreasing function.

Next we introduce the idea of multiple trader types. Let $H$ denote the number of different trader types and let $v_{i+1}^h$ denote the belief of trader $i$ of type $h \in H$ regarding the price of the asset at time $t+1$. Since $v_{i+1}^h = p_t + b_t^h$, and $b_t^h$ is the same for every trader of type $h$, $v_{i+1}^h$ is the same for every trader of type $h$. That is, if we consider traders $i,j$ with $i \neq j$, and both of them are of traders of type $h$ then $v_{i+1}^h = v_{j+1}^h$. Having established this fact, and adding the additional assumption that $y_i^t W_i^t$ are the same for all traders of the same type, we can re-write our equilibrium condition as $\sum_{h=1}^H n_{ht} \frac{y_i^t W_i^h}{v_i^{t+1}} = \sum_{h=1}^H A_i^h$, where $n_{ht}$ represents the number of traders of type $h$ at time $t$. If we impose the further restriction that all traders of the same type always offer the same number of shares for sale, then the equilibrium condition becomes

$$\sum_{h=1}^H n_{ht} \frac{w_i^h W_i^h}{v_i^{t+1}} = \sum_{h=1}^H n_{ht} A_i^h$$

(8)

The assumptions that all traders of the same type, $h$, invest the same amount of wealth or sell the same amount of shares are limiting in the sense that the demand or supply effects of

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16Perhaps a better way to state this is that, if $f(B,s)$, $g(C,s)$ represent immigration in $B$ and $C$ at time $s$, respectively, and if $B$ and $C$ make up the entire market, $N$, and $k(N,s)$ represents the immigration in $N$ at time $s$, then $k(N,s) = f(B,s) + g(C,s)$ and by the sum rule for derivatives $\frac{dk}{dt} = \frac{df}{dt} + \frac{dg}{dt}$. Since the derivatives are represented by the SDE discussed later, we can model immigration for the market if we know immigration for each individual group.
a group are limited – the possibility of agents in a group having a very large endowment that affects equilibrium strangely is eliminated. The assumption that agents of the same type, \( h \), offer the same amount of shares for sale is especially poor as it is very unlikely for people to be offering the same number of shares for sale; even if the agents own the same number of shares they would likely place different quantities for sale. However, we will relax these restrictions later. Working with summations is not ideal since, as we deal with more trader types and eventually consider noise, there are many unknown terms and thus working with the equation analytically is not very tractable. However, given that we can work graphically and numerically, (8) suffices for analyzing the effects of multiple groups being present in the market for the asset, evolution, and also the effect of noise in a one-asset market.

**Multiple groups**

Multiple groups are necessary to make the market interesting. This is because, given our assumptions above, if there is only one group then in period \( t \) every person values the asset at the same price, \( p_t \), and either everybody will demand the good or everybody will supply the good. This does not constitute a market. If we allow noise, however, it is possible that some people will demand the good and some people will supply it, even with only one group. Even in this case, unless the effect of noise is allowed to be very large, valuations will be clustered around group’s shared valuations and therefore the volume will likely not be large. Allowing for multiple groups creates the possibility of a wide range of buy and sell orders, as well as interesting market dynamics. The existence of multiple groups does not guarantee these things though, because it is possible for every group to value the asset at the same price, which would reduce our model to the single-group case. For our purposes we will assume that this never happens.

Before discussing the market equilibrium when there are multiple groups I will quickly summarize our definitions and assumptions regarding (8). \( n_{ht} \) denotes the number of traders of type \( h \) at time \( t \), \( y^h_t, W^h_t \) denoted the fraction of wealth invested in the asset and the total wealth for traders of type \( h \) respectively, and are assumed to be the same for all traders of
the same type. If we allow the wealth of individual participants to vary even among trader
types we will revert to summing over \( i \in N \) and will use the notation \( y^i_t, W^i_t \). \( A^h_t \) denotes
the number of shares offered for sale by traders of type \( h \) at time \( t \). In order to relax some
of these assumptions we will re-write (8) as
\[
\sum_{h=1}^{H} \omega^h_t = \sum_{h=1}^{H} \Lambda^h_t \quad \text{where}
\]
\[
\omega^h_t = \sum_{i=1}^{N} L(i), L(i) = \begin{cases} y^i_t W^i_t & \text{if participant } i \text{ is a trader of type } h \\ 0 & \text{otherwise} \end{cases}
\]
\[
\Lambda^h_t = \sum_{i=1}^{N} K(i), K(i) = \begin{cases} A^i_t & \text{if participant } i \text{ is a trader of type } h \\ 0 & \text{otherwise} \end{cases}
\]
In other words, \( \omega^h_t \) represents the total wealth of type \( h \) traders at time \( t \) and \( \Lambda^h_t \) represents
the total number of shares listed for sale by traders of type \( h \) at time \( t \). Letting \( H = \{a, b, \ldots, h\} \) and expanding the summations yields:
\[
\sum_{h=1}^{H} \omega^h_t = \sum_{h=1}^{H} \Lambda^h_t
\]
Looking at this equation we see that this model presents the same difficulties as the previous
model: this equilibrium condition, expressed in terms of summations, does not tell us much
about the equilibrium or about price movements. One thing that we do know, as a result
of the way the market maker sets price, is that the price has to be between
\( \min v^h_{t+1} \) and
\( \max v^h_{t+1}, h \in H \), but that does not give us much information. Perhaps if the summation
contained a term for the future price, \( p_{t+1} \), the above equation would be more tractable.
Such a term can be placed in the equation by assuming that one of the groups has "perfect
foresight", as defined in (Hommes, 2013). That is, since \( v^h_{t+1} = p_t + b^h_t \), we will assume
that for some \( h \in H, b^h_t = p_{t+1} - p_t \), and therefore \( v^h_{t+1} = p_{t+1} \). This is an unrealistic
assumption, however, for reasons discussed by Hommes (Hommes, 2013). Furthermore,
such an approach yields strange results: for example, if the supply and demand of all
participants except for those with perfect foresight are equal, then we would get a predicted
future price of infinity. We would expect, however, that \( p_{t+1} = p_t \) in such a case, since we
are already in equilibrium, and there would be no reason for someone to buy or sell any
shares if they do not expect the price to change. In order to analyze the above equation,
then, we refer to graphical analysis. This is a lengthy approach as there are many cases, but it yields logical results.

Before beginning our graphical analysis I note that graphical analysis is useful for the model presented here but may not be useful in other models. The reason why graphical analysis is useful for us is due to the way that the market maker sets prices: the market maker sets prices to maximize the number of shares traded which, as previously mentioned, corresponds to the intersection of total supply and total demand. Examining how factors such as investor wealth, amount of shares listed for sale, and changes in price valuations shift or rotate the total demand and supply curves can therefore tell us how these factors impact prices. Our assumption regarding how the market maker sets prices has some ambiguity in it, however. For example, does he take total supply and demand completely as given – that is, does he view all bid/ask orders, create a total demand schedule, and then graph the total demand and supply curves to determine market equilibrium, as in Figure 4 – or does he perhaps use the total demand and supply schedule to extrapolate "full" demand and supply curves – demand (supply) curves that are non-constant below (above) the current price, $p_t$ – as in Figure 5.17 In Figure 4, where the market graphs total demand exactly according the order schedule, the demand (supply) curve is constant below (above) $p_t$ because we assumed that nobody will demand (supply) shares below (above) market price if they expect the price to fall (rise). Furthermore, this assumption implies the assumption of demand and supply being fixed according to the bid/ask order submitted at the end of the

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17For an example order schedule and total demand/supply schedule, see Figure 3
previous period. If this was not the case, then consider what would happen if somebody expects a price increase but the price drops. If the individual’s valuation does not change then they would demand extra shares. Figure 4 does not take this into account. Allowing the market maker to extrapolate the total curves beyond \( p_t \) represents a relaxation of this assumption and yields a more appealing analysis. Figure 5 represents this case. As such, we will briefly discuss the case where total demand and supply are constant at times (Figure 4) and then move on to the case where the market maker extrapolates these curves.\(^{18}\)

To begin our brief discussion of the case where the market maker graphs total demand/supply exactly from information he receives we note that the only part of the total demand and supply curves that we are sure is linear is the constant portion. The functional form of the non-constant part is difficult to specify since allowing different market participants to invest different amounts of wealth or sell different amounts of shares allows the curves to be bumpy. Indeed, this can be seen in Figure 4. The reason for this is that larger increases occur at some prices but not others, specifically when a very wealthy investor decides to demand the asset. We do not make more powerful assumptions here that allow us to use more tractable graphs because in the real world the graphs likely do not follow a nice functional form. Next, we note that the non-constant portion of the supply curve cannot be above \( p_t \) and the non-constant portion of the demand curve cannot be below \( p_t \). This implies that the two curves can only intersect in three ways: the constant portion of the supply curve can intersect the demand curve above \( p_t \); the constant portion of the demand curve can intersect the supply curve below \( p_t \), or supply and demand can intersect at \( p_t \).

The first case corresponds to when total demand is greater than total supply. In this case, the equilibrium will change (specifically, it will be at a higher price) if more shares are demanded at or above the intersection of supply and demand, or if there is any increase or decrease in total supply, which will cause a decrease or increase in the equilibrium price,

\(^{18}\)We assume that the market maker fits a linear function to the demand and supply curves. This is likely not a great assumption, but we make it for the sake of tractability. In the model considered in (Hommes, 2013) "s shaped" supply curves are considered, although for a different market model, which are likely a better approximation of the true shape of supply and demand. Discussing rotations and changes of slope are more complicated when dealing with non-linear curves, however. This is something we will discuss briefly when we analyze the implications when the market maker graphs total demand and supply exactly according to the bid/ask schedule instead of extrapolating them linearly.
respectively. A redistribution of total supply – that is, some group changing their valuation
– will not change the equilibrium, but a buyer changing their valuation from below (above) the
current intersection to at or above (strictly below) the current intersection will increase
(decrease) the equilibrium price. A buyer changing their valuation from below (above) the
equilibrium to another point below (above) the equilibrium will not change the equilibrium
price. An increase (decrease) in a trader types total wealth, $\omega^h_t$, has an effect similar to that
of a decrease (increase) in their valuation, and can be brought about by an increase in the
number of traders of a certain type.

The second case corresponds to when total supply is greater than total demand. In this
case price will increase (decrease) if one or more trader types increases (decreases) their
valuation from at or below (strictly above) the intersection of supply and demand to above
(below) the intersection. Additionally, wealth of suppliers has no effect on $p_{t+1}$, unless we
consider some sort of an endowment effect about the number of shares held and sold. If we
consider an endowment effect, then we would see an effect similar to in the previous model
where a large endowment effect can convert a seller to a buyer, increasing demand and
therefore the price. The total number of shares listed for sale does not necessarily affect the
price, either; what matters is how the shares are distributed among the various valuations.
In the third case, the equilibrium price is only affected if either total supply or total demand
changes. This can happen because of changes in the valuation of a trader type, an increase
in the overall wealth of a trader type – which can be brought about by an increase in the
number of traders of a certain type – or changes in the number of shares offered for sale.

Now we will assume that the market maker does not take the demand and supply curves
as-is but instead extrapolates linear demand and supply curves from the information he has.
That is, we are considering the case of Figure 5. The market maker does still set price at
the intersection of supply and demand. Additionally, this case is quite similar to the above
case, but differences arise since changes even on one side of the intersection will change
the slope everywhere. We will simply provide a quick summary of results here. If the
number of people demanding shares increases, then the equilibrium price will increase. If
the number of people supplying shares increases, then the equilibrium price will decrease.
Both of these results do not depend on which price the increase occurs at, since they will cause either a rotation or a shift of the demand/supply curve. A noteworthy trait of the current model is that if a buyer decreases their valuation then the equilibrium price will increase, all else equal. This is because the slope of the total demand curve will decrease, and is perhaps counterintuitive, but the reason this occurs is because our model states that the equilibrium price depends on the number of shares traded, and a buyer decreasing their valuation means that they will submit a bid for more shares. The way we defined our demand equation is the cause of this, and is a flaw in our model. It can be remedied by assuming that the market maker calculates each market participant’s wealth and uses this information to create a "maximum" demand curve that represents the maximum number of shares that could be sold at any given price, assuming that supply is infinite. Lastly, if a seller increases their valuation then the equilibrium price will increase; if a seller decreases their valuation then the equilibrium price will decrease.

Whether considering extrapolated linear curves or exactly plotted total demand and supply, the number of traders of a certain type can impact price. In order to discuss this we will introduce a stochastic differential equation. The stochastic differential equation will be time dependent, and thus we will not be able to solve it, but we examine the deterministic case and present a simulation of the time-dependent case.

**Introducing Evolution**

**Number of Noise Traders**

Now that we have spent some time describing the model I move on to attempting to characterize the number of traders in a group at time $t$, as well as how this number changes over time. This is a meaningful task since the number of traders in a group has ramifications for the future price – if a group values the asset at an unusually low or high price and has many members they will impact the market price.

A fact worth noting before jumping into the mathematics is that this section is less economics and more biology; noise traders are a population and the growth and decline
of the population should therefore be able to be modeled. While this section will likely
not introduce concepts that are new to biology or even finance, the idea of modeling group
population as it evolves is, as far as I know, an application of biology and evolution to
finance that is unique to this paper. It is also an idea I am now emphasizing because I
believe modeling how group population evolves is something that should become standard
practice in the examination of market efficiency and asset pricing, and also the fact that
doing so uses concepts of evolutionary biology, thus the concept agrees with the adaptive
market hypothesis. Granted, modeling group population may not be a prominent feature of
papers because doing so in practice is likely difficult. Indeed, the model I construct I am
not able to solve. However, I solve it for a simplified, deterministic case, and I numerically
analyze the model. Such an approach is sufficient for my goal of introducing the idea to the
field. Lastly, I use stochastic differential equations in this section (which I may abbreviate
SDEs in the future), and I refer to some stochastic financial models. The reason why
stochastic differential equations are appropriate for this task is that sometimes people enter
or leave the market for exogenous reasons. For a review of or a reference to the concepts
used in this section see (Øksendal, 1998) and (Wilmott, 2007).

The key concept for this section is that groups, and individuals within groups if we allow
noise, differ. Considering the noisy case, how do individuals differ? The obvious answer
is that they differ because their noise differs. We need a way to model this difference,
though. We can do so by realizing that the difference between two noisy traders lies in how
accurately they value the asset; some people are noisier than others. In order to characterize
the fact that some investor’s are noisier than others I use an approach similar to Luo (Luo,
2012, p. 66). My approach differs in that I add indexes for time since I do not assume that
the probability distribution of a speculator’s predictive ability is identically distributed over
time. Therefore, let

\[ \Omega^i_{1t} = Pr(v^i_{t+1} > p_{t+1}) \]
\[ \Omega^i_{2t} = Pr(v^i_{t+1} = p_{t+1}) \]
\[ \Omega^i_{3t} = Pr(v^i_{t+1} < p_{t+1}) \]

where superscript \( i(j) \) indicates buyer (seller) \( i(j) \) and \( \Omega^i_{1t} + \Omega^i_{2t} + \Omega^i_{3t} = 1 \).\(^{19}\) Each
market participant, when they enter the market at time \( s \), is characterized by the vec-

\(^{19}\)This is the same as saying \( \Omega^i_{1t} = Pr(\xi_t > 0) \mid \Omega^i_{2t} = Pr(\xi_t = 0) \mid \Omega^i_{3t} = Pr(\xi_t < 0) \)
tor \((\Omega^i_{1s}, \Omega^i_{2s}, \Omega^i_{3s})\). This vector characterizes the probability distribution of participant \(i\)'s over-prediction, exact prediction, and under-prediction at time \(s\) – in other words the probability distribution of his predictive ability – and evolves over time for all traders. The reasoning behind having this distribution evolve over time as opposed to remaining fixed as in (Luo, 2012, p. 66) is that, while Luo states that "speculators display systematic bias due to cognitive errors" I believe that he fails to take into account that the source of those cognitive errors may change over time. The endowment effect discussed earlier is an example of a cognitive bias changing over time; investor holdings change frequently and therefore the endowment effect should as well. The availability heuristic will change over time as well, although less frequently than the endowment effect. The availability heuristic will change because investors can experience a period of high returns or a period of low returns. Whatever period they experienced most recently will be weighted heavier in their mind and influence their trading. If we were to assume that some investors are not subject to cognitive bias then the vector \((\Omega^i_{1s}, \Omega^i_{2s}, \Omega^i_{3s})\) would be fixed for those traders.

One other use of this vector characterizing the probability distribution of some predictive ability is to estimate the predictive ability of the market at time \(t\), where the predictive ability of the market is the average predictive ability taken over every participant in the market. Letting \((\overline{\Omega}^i_{1s}, \overline{\Omega}^i_{2s}, \overline{\Omega}^i_{3s})\) denote the average predictive ability of the market at time \(s \in T\), \(\overline{\Omega}^i_{1s}\) is the probability of the market over-predicting the future price, \(\overline{\Omega}^i_{2s}\) is the probability of the market predicting the future price correctly, and \(\overline{\Omega}^i_{3s}\) is the probability of the market under-predicting the average price. These averages are simply the arithmetic mean of their respective probabilities, taken over the total number of traders in the market. The idea of the probability of the market predicting the correct future price is useful if one wants to consider learning. If there is a way to track the average predictive ability of the market, then we could get an idea of how \((\overline{\Omega}^i_{1s}, \overline{\Omega}^i_{2s}, \overline{\Omega}^i_{3s})\) changes over time. This would allow us to speculate about how individuals' \((\overline{\Omega}^i_{1s}, \overline{\Omega}^i_{2s}, \overline{\Omega}^i_{3s})\) change over time. It would also let us get an idea about whether investors are "dying out" and leaving the market, or if new investors are simply better than old investors.

So, how does vector \((\Omega^i_{1s}, \Omega^i_{2s}, \Omega^i_{3s})\) evolve? In this model it is assumed that the change
in an individual’s predictive ability (in other words the evolution of the above vector) occurs solely as the result of learning. Three types of learning are examined in here: bad learning, good learning, and unknown learning. Bad learning describes the case when \( \lim_{t \to \infty} (\Omega_{1,1}^t, \Omega_{2,1}^t, \Omega_{3,1}^t) = (x, 0, y), x, y \in [0, 1], \forall i \in \mathcal{L} \) and \( x + y = 1 \). In other words, bad learning is when all traders become more likely to incorrectly predict the future price as time passes.\(^{20}\)

This could be the case if investors overreact, as they will underuse losing strategies and overuse strategies that have won in the past, even if the strategy is not fit for the current time period.\(^{21}\) It could also be the case if we assume that investors never become better at predicting the correct future price then we get the case of bad learning. Bad learning is an unlikely scenario because, over time, investors should learn to predict the correct price, or at least to generate incorrect predictions less often. This can be explained by incentives; learning to not wrongly predict the future value of the asset is fueled by the incentive to not lose money. Furthermore, learning to correctly predict the future price is fueled by the desire to make money. If an investor does not stay in the market long enough – if they do not survive – then bad learning will be a decent estimation. This is because an investor that doesn’t stay in the market long will not have time to learn. However, if an investor does not die out but simply takes a break from the market, what happens when he returns? Will his predictive distribution be the same as when he left or worse? Will he start learning when he returns? These are interesting questions, and seeing as how the survivorship literature has not reached a consensus regarding whether investors die out or not, these questions may be worth researching. It is not my goal to reach conclusions regarding this idea of bad learning, but one way to do so would be by defining parameters such as the probability of the speculator thus overreacting – perhaps due to losing money – and examining how this affects the expected values of a speculator’s \( \Omega \)’s. I do not do so because here we are simply interested in how the group population changes when \( \Omega_{2,i}^t \) does not increase. Good learning is the opposite of bad learning: As \( t \to \infty \), \( \Omega_{2,i}^t \to 1, \forall i \in \mathcal{L} \), and no assumptions are made here regarding overreacting or underreacting. What good learning implies is that

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\(^{20}\)Equivalently, all noise traders become less likely to correctly predict the future price as time passes.

\(^{21}\)(Lo, 2004) argues that this behavior is what leads to irrational behavior – investors may not realize that their strategy is not fit for the current market environment.
traders benefit from their experiences over time and become better at making predictions. Unknown learning is the case between bad and good learning, where $\Omega_{2t}$ may or may not converge to 0 or 1 as time goes to infinity. Unknown learning is therefore the most general case and is likely the best model of real noise trader behavior of the three. It is also the most complicated to model.

Next we move onto actually modeling the number of noise traders over time through the use of stochastic differential equations. The first case we consider is that of bad learning. In a standard differential equation the population would be something like $\frac{dN}{dt} = \alpha N + c$ where $N$ denotes the size of the population, $\alpha$ denotes a rate of growth (or a rate of decline) and $c$ denotes some constant immigration (or emigration). What happens when the rate of immigration is not constant and cannot be specified? For example, what if immigration depends on the environment? This is handled through the use of a noise term. Incorporating the noise term means replacing $c$ with $c + W_t$, $W_t$ white noise. Our differential equation now becomes $\frac{dN}{dt} = \alpha N + (c + W_t)$, a stochastic differential equation.\(^{22}\) I define $\alpha$ as $f(\Omega_{2t}, z_t, \bar{y}_t)$ where $\Omega_{2t}$ is the arithmetic mean of $\Omega_{2t}$ taken over all traders , $z_t$ is a variable denoting the strength of the market at time $t$, and $\bar{y}_t$ is the average fraction of wealth invested by noise traders in the asset at time $t$. $\Omega_{2t}, \bar{y}_t$ affect the rate of decline of noise traders because the more wealth they invest on average, the more they are wrong on average, the more people will leave on average. $z$ is included as a state variable because whether the market is bearish or bullish can affect how well traders perform.\(^{23-24}\) Our SDE now looks like

$$\frac{dN}{dt} = f(\Omega_{2t}, z_t, \bar{y}_t)N + (c + W_t).$$

Taking our differential equation, $\frac{dN}{dt} = f(\Omega_{2t}, z_t, \bar{y}_t)N + (c + W_t)$ and multiplying both sides by $dt$ yields

$$dN = f(\Omega_{2t}, z_t, \bar{y}_t)N \, dt + c \, dt + dB_t$$

\(^{22}\)Note that $c$ and $W_t$ should not be combined because, as shall be seen soon, $c$ likely depends on more than just $t$.

\(^{23}\)Perhaps a noise term should be included in $\alpha$. That is, perhaps $\alpha = f(\Omega_{2t}, z_t, \bar{w}_t) + W_{t_1}, W_{t_2}$ white noise. This is not considered for the sake of simplicity.

\(^{24}\) $\Omega_{2t}$ discusses how often noise traders are right on average, but can be turned into a measure of how wrong people are on average through some method such as taking $1 - \Omega_{2t}$. 

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Noting that $W_t \, dt = dB_t$ where $B_t$ is standard Brownian motion,\(^{25}\) and that $f$ is time-dependent. Re-writing (9) one more time as $dN = [c + f(\Omega_{2t}, z_t, y_t)N] \, dt + dB_t$ makes it clear that our SDE resembles the Vasicek model of the short-term interest rate, $dr = (\nu - \mu r) \, dt + \sigma dX$ where $dX$ is a Brownian increment, with $\nu = c, \mu = -f(\cdot)$ and $\sigma = 1$.\(^{26}\) However, in the Vasicek model $\mu$ is not time dependent. I am unsure if there is a model similar to the Vasicek model but with a deterministic $\nu$ and time-dependent $\mu$. The Hull-White model has a time-dependent $\nu$ and a deterministic $\mu$, and the Extended-Vasicek model pertains to both $\nu$ and $\mu$ being time-dependent. We can extend our model to the case of a time-dependent $\nu$ and $\mu$ by making the amount of immigration depend on time. A simple way of doing this is by making $c$ depend on the state of the market, $z$. Therefore, let $c = g(z_t)$. Re-write our differential equation one more time as

$$dN = [g(z_t) + f(\Omega_{2t}, z_t, y_t)N] \, dt + dB_t$$

(10)

Presenting analytical solutions to this SDE is beyond the scope of this paper,\(^{27}\) but graphs of the deterministic case are presented. We also present an attempt at graphing the time dependent case. This SDE represents not only the case of bad learning that we used to introduce the SDE, but also good learning and unknown learning. The difference between the three cases in the SDE is that for bad learning $\left(\frac{\partial f}{\partial \Omega_{2t}}\right)_{y_t,z_t}$ increases with time (it becomes more negative, since $f(\cdot)$ describes noise traders leaving the market) whereas for good learning it decreases with time (noise traders become smarter and therefore fewer leave the market as a result of generating incorrect predictions) and for unknown learning it may either increase or decrease with time.

The last fact I wish to point out before presenting the graphs is that $c = g(z_t)$, as defined above, must represent both immigration and emigration. This is because $c$ is where noise was introduced to our differential equation, and a sufficiently large negative noise term could produce a negative value of $c$. A negative value of $c$ would represent negative

\(^{25}\)See (Øksendal, 1998)
\(^{26}\)See (Wilmott, 2007, Page 134)
\(^{27}\)For the solution to the simplified case where $g(\cdot)$ and $f(\cdot)$ are deterministic, see appendix B.
immigration – the same thing as emigration. If one wants to restrict \( c \) to representing immigration and include emigration in \( f(\cdot) \) it is possible to redefine \( c \) as \( c = d \times N \) where \( d \) is the rate of immigration. Since rates are allowed to be negative, introducing noise into \( d \) is not questionable. Making such a substitution changes our SDE to

\[
dN = [g(z_t) + f(\Omega_2, z_t, w_t)]N \, dt + N \, dB_t
\]

The following graphs are for 10. Graphs of 11 are not presented. Note that in the following graphs, \( \sigma \) refers to \( f(\cdot) \).

**Figure 6: Deterministic SDE Graph 1**

Figures 6,7,8,9 are graphs of how the noise trader population may change over time, assuming that \( g(z) = c \) and \( f(\cdot) \) are not time-dependent. All four graphs have an initial population of 1,000,000 noisy traders. \( \sigma \) at the top of these figures refers to \( f(\cdot) \) in our equation. In Figure 6 immigration, \( c \) is much greater than the rate at which noise traders exit, \( f(\cdot) \), and therefore the noise trader population increases over time. In Figure 7 the opposite is occurring; noise traders are initially leaving much more quickly than they enter the market. The number of noise traders does not go to zero, though. A steady state is reached at about \( N=200,000 \), where the rate at which noise traders leave is about the same as the rate at which they enter. This indicates that arbitrageurs may not be able to
eliminate a noise trader population, unless the arbitrageurs are overwhelmingly powerful (the arbitrageurs would need to eliminate noise traders at a rate much faster than noise traders enter the market). In figure 8 noise traders are leaving and entering the market at about the same rate; this graph is driven by the Brownian motion. If the Brownian motion has a mean of 0 then the number of noise traders will tend to stay around the initial value, although this does not have to be the case – sometimes they decreased greatly and other
times they didn’t.

In Figure 9 \( f(\cdot) = 1 \), which means that all noise traders leave every period. This is interesting because it shows that the noise trader population still persists. The noise traders in the market persist due to the entry of new noise traders into the market. While my noise term may have been too large, thus making the number of noise traders entering the market seem unusually large, even without noise the noise trader population would continually re-emerge because of the immigration term – this process reverts to the number of noise traders entering the market at a given time. This argues against the presence of arbitrageurs causing markets to be efficient – they may make markets efficient, but they cannot keep them efficient. The empirical question then becomes: for what length of time are markets efficient? Also, it is possible that once the noise trader population dies out new noise traders will not enter. This could be the case if noise traders enter the market based on some fitness measure. We will discuss fitness measures later, but if noise traders stop entering the market once the initial population dies off, then arbitrageurs can indeed eliminate noise traders. These four figures were modeled in Mathematica using the ItoProcess feature, a Wiener Process to model the noise term, and ListLinePlot to plot the graphs. Figure 10 is

Figure 9: Deterministic SDE Graph 4

![Figure 9: Deterministic SDE Graph 4](image)

...the most interesting picture. This picture corresponds to a simulation of the time-dependent
stochastic different equation. This simulation was performed in excel. It was constructed by generating a Wiener process in Mathematica, in order to get the $dB_t$ terms, randomly generating parameters for $w$ and $z$, $z$ taking on values between $-1$ and $1$ to represent the strength of the market at time $t$, and randomly choosing $c$ from $\{-10000, 10000\}$, so $c$ correctly represents both immigration and emigration. The interesting features of this graph are that the noise trader population does indeed die out. However, it keeps resurfacing, and for quite a few time periods – the full horizontal scale is 5000 time periods. This says that noise trader populations will continue resurfacing in a market. Also worth noting is that the rate at which the noise trader population dies out, and even the fact that it dies out depends on the parameters used. Sometimes the noise trader population trends around a steady state value that is not zero. The conclusion that we reach is therefore that arbitrageurs do not necessarily force noise traders out of the market. They can, and sometimes they might, but it is not guaranteed.
Continuing With Evolution

In order to introduce evolution into our model all we have to do is incorporate our SDE into it. We can do this by using our SDE to specify $n_{ht}$, the number of traders of type $h$ at time $t$. Incorporating the SDE in such a way will allow $n_{ht}$ to update every time period according to a form of natural selection. This is what will make this model evolutionary. All we need to do to incorporate our SDE into this model is to specify an SDE for each group, $h$, and then replace $n_{ht}$ with the appropriate SDE. This is a simple concept, so instead of discussing it in depth this section will compare our SDE to Hommes’ evolutionary model created through the specification of a performance measure.

The performance measure used by Hommes is evolutionary fitness. The fitness of a population is something that was discussed by Darwin in the 19th century; it is not a new idea. It has even been applied to mathematical models before, such as in (Nowak, 2006). The idea of fitness is not new to economics, either. Hommes uses the idea of fitness slightly differently than the standard usage in the field of Biology, however. Hommes uses fitness as a synonym for a performance measure: the fitness of a trading type is an indication of how well that trading type performs. It is not a direct indication of survival. This is similar to relative fitness in Biology, which compares the average number of survivors of a species to the average number of survivors of all species in a generation. There is nothing wrong with Hommes’ approach; naturally, people will trend towards the trading type with the highest relative fitness. I prefer to take a different approach, however. In fact, our stochastic differential equation already includes a measure of fitness.

The fitness measure in (10) is a measure of absolute fitness, not relative fitness. Absolute fitness is essentially the ratio of survivors of a species to the total number of members of the species at the start of the time period. Mathematically, $\Phi_{abs} = \frac{N_{after}}{N_{before}}$. Rearranging this, we get that $\Phi_{abs} \cdot N_{before} = N_{after}$. When we derived our differential equation, we said that $f(\cdot) \cdot N$ represents the number of people who leave the population. Therefore, $f(\cdot) \cdot N = N_{before} - N_{after} = N_{before} - \Phi \cdot N_{before} = N_{before} \cdot (1 - \Phi)$ which implies that $f(\cdot) = (1 - \Phi)$, one minus the absolute fitness. As far as I know there is not a term
for one minus the absolute fitness, however the existence of this term in 10 indicates the relationship between our SDE, absolute fitness, and evolution.

At this time I wish to make a slight modification regarding (10). Earlier we stated that immigration/emigration is represented by $c$ in the deterministic model, $g(z)$ in the time-dependent model. Having introduced the idea of absolute fitness, however, we can change $g(z)$ to depend on absolute fitness and market strength. Fitness, absolute or relative, as a determinant of immigration is a must since people will gravitate towards the best-performing trader types. Conversely, people will leave the worst performing groups. This is a consequence of seeking the highest expected returns. We leave $g$ dependent on market strength, $z$, as well because immigration is more likely when the market is strong and emigration is more likely when the market is weak. Letting $\mathcal{A}_t$ represent absolute fitness at time $t$, (10) becomes

$$dN = [g(z_t, \mathcal{A}_t) + f(\Omega^2_t, z_t, w_t)N]dt + dB_t.$$ Note that $\mathcal{A}_t$ can be replaced by $1 - f(\cdot)$.

In addition to Hommes using relative fitness and me using absolute fitness, Hommes and I evaluate fitness using different metrics. Hommes discusses using accumulated realized profits or risk-adjusted profits as fitness measures. My fitness measure also touches upon profits, but is more comprehensive. In my SDE profits are reflected by the $\Omega^2_t$ term in $f(\cdot)$ However, I believe that $f(\cdot)$ is the additive sum of $(1 - \Omega^2_t) \cdot w_t$ and $z_t$, likely with some multiplicative constants. The reasoning behind this is that $(1 - \Omega^2_t)$ represents the probability of being wrong, and the more wealth people invest and the more likely they are to be wrong then the more likely they are to not survive the period. Market strength also matters because people should perform worse when the market is down and vice versa.

Hommes’ use of fitness to define the fractions of each trader type is powerful because it provides a clear indication of how traders will be distributed, and it is a more tractable approach than using stochastic differential equations. Even so, using SDEs is a more comprehensive approach. Both approaches are able to compensate for immigration; the SDE approach developed here does it directly while using a fitness measure accounts for it through the addition of a noise term. Using an SDE, however, will better model behavior regarding strategies with low fitness. If one makes a prediction just based off of a fitness measure, a
strategy with near-zero fitness should have almost no traders choose it. However, a strategy may develop a low fitness during a certain state of the economy or the market and then become desirable when the market changes. SDEs can model this by having a large amount of immigration when the fitness measure becomes desirable, and then the fitness can be increased if the strategy indeed performs well. A stochastic differential equation such as the one above can also predict this immigration because the state of the market can be incorporated. In the SDE we derived we only have a variable for the strength of the market, but we could be more specific regarding whether it is bearish or bullish, bullish becoming bearish, etc. We could also incorporate the strength of a sector and the conditions under which a strategy is desirable. The downside to doing so is that the tractability of the model would be reduced.

**Considering The Possibility of Noise**

This section is a brief consideration regarding if we allow the valuations of traders of the same type to vary by a noise term. That is, let \( v_{t+1}^{hi} = p_t + b_t^h + \epsilon_t^i \) be individual \( i \)'s valuation of the asset at time \( t \). The \( \epsilon \) term means that it’s possible for no two traders to have the same valuation at time \( t \). One possible way to analyze the effect of noise on equilibrium would be to define the mean and variance of \( \epsilon \), and then to look at the distribution of valuations in order to get an idea of how people will value the asset at a certain time. This could then be used in our demand and supply equations. Using the earlier demand and supply equations, however, leads to the same problem of not being able to analytically examine equilibrium. We could once again use graphical analysis but the results would be the same as earlier – incorporating noise will not change the effect any variables had. The presence of noise just means that the variables may not all shift by the same amount. I believe that the best way to look at the effect of noise would be numerically, which is currently underway. Since I cannot do analytical analysis, and I am not done with my numerical analysis, I will simply devote a brief paragraph to discussing the possible effects of noise.

The most important thing to note about the noise is that a small bit of noise can move
the market out of equilibrium. Consider the linear function extrapolation model: a small amount of noise for one person would change the slope of a curve and thus change the equilibrium. It also must be noted, however, that two equal and opposite noise terms (for different participants) would perhaps restore equilibrium. That is, in our model it is easy to leave equilibrium, but being in equilibrium is also possible. Which of the two is more likely depends upon one's belief regarding how \( \epsilon \) is distributed across all noise traders.\(^{28}\) If there are many noise terms, our demand schedule will be more "bumpy". Under the linear extrapolation model the noise may not have a large effect, but under the as-is models noise near equilibrium could be affect equilibrium quite a bit because the slope depends on every point. Earlier we stated that the market maker extrapolating a "maximum demand curve", could help remedy this issue of slope dependence. This suggestion would also lesson the affect of noise.

One last concept I wish to introduce is that people will know that they are noisy, and they will therefore hedge against this. One way in which we can represent this is through the use of the vector \((\Omega_1^t, \Omega_2^t, \Omega_3^t)\). However, doing so would not be realistic as people do not know their own probability of failing to correctly predict the future value of the asset. They can estimate this vector, however. Or they could at least estimate the average over time of this vector. They way they could do this would be by looking at their past returns. More specifically, they could look at the deviation of their predictions from the realized price over past periods. This approach is still somewhat flawed since the source of their incorrect predictions could be their trader types, but over time and over many different trader types an individual should be able to get some sort of idea of how right or wrong they usually are. Additionally, since everyone is noisy an individual could compare their past predictions to the past predictions of traders who were the same type in order to get a better idea about ones possibility of misprediction. Figuring out the type of trader other individuals were would be difficult to figure out, but bid/ask data are accessible, and many people who invest a lot interact with other investors, so such a comparison is a possibility. What would be the

\(^{28}\)Note that for our model this distribution should not be normal. That is because if the \( \epsilon \)s are normally distributed, \( E(\frac{1}{\epsilon}) \) is not defined, and such expectations may arise in our model
As stated a few lines earlier, this information can be used to hedge against oneself. For example, if you know that you often overpredict price changes, you can use this to limit how many shares you buy or sell. If for investor $i$ $Ω^i_3 = .5$ then he overpredicts half the time. If investor $i$ knows this then he may use this information to regulate his own trades. For example, he may only buy/sell a fraction of the shares he would buy/sell if he were always right. Assume that he adjusts all his orders by half of his probability of overprediction. That is, assume that all his trades are adjusted by $\frac{5}{2} = .25$. If he was going to sell 100 shares he would keep 25; If he was going to buy 100 shares he would only buy 75. He partakes in this behavior just in case he makes a bad prediction. What kind of effect would this have on the market? What kind of effect would this have on his wealth? Perhaps this would reduce the total amount of noise in the market, if every individual did such a thing. Such a behavior is something to consider in future models of the market.

**Conclusion**

In the paper we established a framework for examining market equilibrium. We then applied the framework to two simple models. We found that, if an endowment effect exists in asset markets, it will increase the price of an asset, and the presence of an availability heuristic can either increase or decrease an asset’s price. Cognitive biases such as these, when present at the same time, may counteract each other as in Figure 2. However, they may also cause a redistribution of wealth to people who are noisier, which will make the future price noisier since biased traders will be able to buy or sell more shares. This redistribution of wealth can occur because a cognitive bias – a type of noise – does not necessarily lead to losing money; it can be favorable in the short run and result in an increase in wealth. In the long run, though, a biased trader may indeed lose more than they win and thus be forced to exit. If the long run is considered, however, the fact that the population of biased traders may increase must be considered; biased traders are continually entering the
market. Figures 6 through 10, which are graphs of the SDE developed to model the population of a group of traders over time, illustrate this fact. From the SDE and graphing it for different values it is clear that noise traders can be competed out of the market, even in the long run. This is what almost what happens in Figure 7. Figure 7 reaches a steady state around 200,000 noise traders, though, because only 25 percent of noise traders leave the market each period – whether they go to a different market or die out is not specified. As the number of noisy traders decreases the number that leave decreases and eventually equals the level of immigration. Figure 9 shows what happens if arbitrageurs are able to eliminate all noise traders every period. This graph makes it clear that even in this case noise traders can still persist through immigration. Figure 9 is exaggerated though, as the noise is on the order of around 100,000 noise traders. Figure 10, which simulates the time dependent stochastic differential equation – versus assuming deterministic coefficients as Figure 6-9 do – agrees with Figure 9; noise traders die out but more noise traders enter. What these figures indicate is that perhaps the question should not be whether noise traders leave the market but how long it takes arbitrageurs to make noise traders leave the market. If arbitrageurs do so fairly quickly markets should be quite efficient. If they do so slowly then figures 9 and 10 suggest that the markets usually have quite a few noise traders.

The framework developed in this paper, while not very tractable when applied to the models presented alongside it, has a lot of room for development. Its selling point is also that it is flexible. In the future it can be applied to other kinds of assets such as forex and futures. Short selling can be incorporated, as well as orders other than limit orders. Behavior such as buying and selling at the same time can be added as well, and consideration of non-linear demand and supply curves may be possible. Most importantly, a solution should be found for time dependent stochastic differential equation, the model can be applied to empirical data, and many other methods of predicting the future value of the asset can be considered.
References


Appendix A

**Theorem 1.0** The market maker setting price in order to maximize the number of shares traded implies that he sets the price corresponding to the intersection of supply and demand:

**Proof Sketch:** Due to the nature of supply and demand, when price is high $Q^d$ is low and $Q^s$ is high. Conversely, when price is low $Q^d$ is high and $Q^s$ is low. Since $Q^d$ is strictly increasing and $Q^s$ is strictly decreasing, once they intersect they cannot cross again. This trait, combined with the fact that the quantity traded is $\min_{Q^d, Q^s}$ implies that the market maker, since he sets price to maximize the number of shares traded, will set a price corresponding to this intersection.\(^{29}\)

Appendix B

Solution to equation 9 if $f(\cdot)$ and $c$ are assumed to be deterministic:

We begin with equation 9, $dN = f(\Omega_{2,t}, z_t, w_t)N dt + c dt + dB_t$. Rearranging terms we arrive at $dN - f(\Omega_{2,t}, z_t, w_t)N dt = c dt + dB_t$. Then, by multiplying both sides of the equation by $e^{f(\cdot)t}$ we get $e^{f(\cdot)t}dN - f(\cdot)t e^{f(\cdot)t}N dt = ce^{f(\cdot)t}dt + e^{f(\cdot)t}dB_t$.

The next step is realizing $\frac{d(e^{f^c(t)}N)}{dt} = e^{-f^c(t)}dN - f(\cdot)t e^{-f^c(t)}N$, so $\frac{d(e^{f(\cdot)t}N_0)}{dt} = ce^{-f(\cdot)t} + e^{-f(\cdot)t}dW_t$. Using this information we can write $d(e^{-f(\cdot)t}N_t) = ce^{-f(\cdot)t}dt + e^{-f(\cdot)t}dB_t$. Next we use integration to get that $e^{-f^c(t)}N_t = N_0 + \int_0^t e^{-f^c(s)}c ds + \int_0^t e^{-f^c(s)}dB_s$.

The final steps consist of simplifying the equation. We move the exponential term to the right side of the equation to get $N_t = N_0 e^{f(\cdot)t} + c \int_0^t e^{f(\cdot)(t-s)} ds + \int_0^t e^{f(\cdot)(t-s)} dB_s$, and then simplifying the integral gives us the final result:

$$N_t = N_0 e^{f(\cdot)t} - \frac{c}{f(\cdot)} (1 - e^{f(\cdot)t}) + e^{f(\cdot)t} \int_0^t e^{-f(\cdot)s} dB_s$$

\(^{29}\)In short $\forall$ price, $p \mid p \not= p_{eq}$ where $p_{eq}$ is the price corresponding to equilibrium, one of the curves will be at a quantity below the equilibrium quantity. Since $p_{eq}$ yields the equilibrium quantity, the market maker will make $p = p_{eq}$ in order to maximize the quantity traded.