Use of the Monte Carlo Simulation in Valuation of European and American Call Options

Gorica Malesevic

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Abstract
This thesis examines the valuation methods used for pricing European and American call options. Options are financial instruments that play an important role in the financial industry and are used in hedging, speculating and arbitraging. Because options are widely used in investing, there is a need for valuation methods that are as precise as possible. Options have been perceived as obscure financial instruments due to the lack of valuation techniques in the past. However, with the discovery of Black-Scholes Model in 1973, the first option valuation method, option trading escalated. In this thesis, the fair market value of S&P 500 index with European exercise style, The Google Option Contract and Apple Option Contract will be obtained by using the Black-Scholes Model, the General Monte Carlo Simulation, The Combined Method and the Least-Squares Monte Carlo. The results from three models will be compared and contrasted in order to determine the best valuation method.

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LAKE FOREST COLLEGE

Senior Thesis

Use of the Monte Carlo Simulation in valuation of European and American call options

by

Gorica Malesevic

April 25, 2017

The report of the investigation undertaken as a Senior Thesis, to carry two courses of credit in the Department of Economics, Business and Finance

Michael T. Orr
Krebs Provost and Dean of the Faculty

Carolyn Tuttle, Chairperson

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Abstract

This thesis examines the valuation methods used for pricing European and American call options. Options are financial instruments that play an important role in the financial industry and are used in hedging, speculating and arbitraging. Because options are widely used in investing, there is a need for valuation methods that are as precise as possible. Options have been perceived as obscure financial instruments due to the lack of valuation techniques in the past. However, with the discovery of Black-Scholes Model in 1973, the first option valuation method, option trading escalated. In this thesis, the fair market value of S&P 500 index with European exercise style, The Google Option Contract and Apple Option Contract will be obtained by using the Black-Scholes Model, the General Monte Carlo Simulation, The Combined Method and the Least-Squares Monte Carlo. The results from three models will be compared and contrasted in order to determine the best valuation method.
Acknowledgments

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Chapter 1: History of Option Trading

Due to the fact that options are an important segment of the financial markets, it is important to get a better understanding of them. Options are used by many investors on a daily basis as tools of hedging, speculating and arbitrage. Since options play important role in financial markets and the economy in general, it is crucial to ensure the proper valuation of their fair value.

This paper describes the analytical and mathematical tools used in the process of estimating the fair value of European and American call options. This paper is focused on three valuation cases: 1) Valuation of American Call Option with no dividends (GOOGLE); 2) Valuation of European Call Option with dividends (SPX); and 3) Valuation of American Call Option with dividends (APPLE).

The Black-Scholes-Merton Model, The General Monte Carlo Method, The Monte Carlo Method on The Black Scholes, and Least-Squares Monte Carlo Method are used in order to valuate call options. Those four methods are implemented by Python in order to accelerate and simplify the estimation procedure.

This thesis is organized in five individual chapters. The first chapter includes the history and development of options. The second chapter gives a brief overview of option terminology that is crucial for the understanding of this topic. In the third section The Black –Scholes-Merton and the General Monte Carlo theoretical valuation models are presented. Chapter four is the literature review focused on option valuation. In the Chapter five empirical work and results are presented and conclusion is made.
This chapter gives a brief overview of the historical development of option trading. It begins by examining the option trading in Greek civilization in 4th Century BC. It also describes Tulipmania, an option trading event in Netherlands that had caused many Dutch investors to lose their wealth and pushed the Dutch economy into the recession. For many years, investors have been skeptical about option trading since they realized the consequences of Tulipmania. Discovery of The Black Scholes Pricing Model, however, allowed investors to estimate theoretical option prices has raised investors’ confidence. The increase in investors’ confidence had triggered the expansion of option markets.

Options trading can be traced back to Greek civilization. Through the work *Politics*, the Greek philosopher Aristotle provided a reference about the successful speculation of the philosopher Thales. Although Aristotle’s short anecdote is lacking information related to the nature of the contract, it shows the existence of informal option contracts in the past (4th Century BC).

There is, for example, the story, which is told of Thales of Miletus. It is a story about a scheme for making money, which is fathered on Thales owing to his reputation for wisdom; but it involves a principle of general application. He was reproached for his poverty which was supposed to show the usefulness of philosophy; but observing from his knowledge of meteorology (so the story goes) that there was likely to be a heavy crop of olives [next summer], and having a small sum at his command, he paid down earnest-money, early in the year, for the hire of all the olive-presses in Miletus and Chios; and he managed, in the absence of any higher offer, to secure them at a low rate. When the season came, and there was a sudden and simultaneous demand for a number of presses, he let out the
stock he had collected at any rate he chose to fix; and making a considerable fortune he succeeded in proving that it is easy for philosophers to become rich if they so desire, though it is not the business which they are really about (Poitras, 2013, p.57).

The interesting event that illustrates the use of options in the modern era was the Tulip Bulb Mania, also known as Tulipmania in the 17th Century. This event occurred in Holland where tulips had high popularity and have been a symbol of Dutch aristocracy. Tulipmania had its roots in the Ottoman Empire when Suleiman the Magnificent noticed the flower in the 15th Century (Holodny, 2014). Tulips popularity had spread across Europe and tulip demand increased significantly. The increase in demand for tulips caused the increase of their price, which served as an incentive for Dutch families to start investing in tulip contracts using their savings or borrowing against the assets that they possessed (ex. homes). In the article published in The Economist (October 2013 edition), historical data were used to show the irrational value of rare tulips:

At the beginning of 1637, some tulip contracts reached a level about 20 times the level of three months earlier. A particularly rare tulip, Semper Augustus, was priced at around 1,000 guilders in the 1620s. But just before the crash, it was valued at 5,500 guilders per bulb—roughly the cost of luxurious house in Amsterdam (Aievoli, 2016, p.9).

Tulip growers used to buy puts to protect their profit in case that the price of tulip bulbs decreased. Tulip wholesalers would buy calls to protect themselves of the risk caused by decreases in tulips’ price. The price of tulips continued to rise and eventually it reached the point when the “bubble burst” and the obligations for that season’s bulbs became worthless (Goldgar, 2008). Since Dutch people had invested everything that they had in
tulips, the crash of the tulip market made them lose their money and homes. Due to the fact that the tulip option market was unregulated, it was impossible to make investors fulfil their obligations specified in their option contracts. In the book *Tulipmania: Money, Honor, and Knowledge in the Dutch Golden Age*, Anne Goldgar presents two important myths of Tulipmania. She argues that the drop in prices wiped out the merchant class. Additionally, Goldgar claims that the Dutch economy went into a recession and options gained a negative reputation globally.

In the period between the 16th and 19th Century, options were banned in many countries since people had understanding of Tulipmania debacle. Options were banned across the world: from Europe, Japan, and some states in the US but the most notable ban was in London (England). In 1733, Sir John Barnard introduced the bill *An Act to prevent the infamous Practice of Stock-jobbing in the UK* in order to prevent option trading. The main provision of Barnard’s Act was the following:

All contracts or agreements whatsoever by or between any person or persons whatsoever, upon which any premium or consideration in nature of a premium shall be given or paid for liberty to put upon or deliver, receive, accept, or refuse any public or joint stock, or other public securities whatsoever, or any party, share or interest therein, and also all wages and contracts in the nature of wagers, and all contracts in the nature of puts and refusals, relating to the present of future prices or value of any stock or securities, as aforesaid, shall be null and void (Poitras, p.32).

It is interesting to add that the ban in London lasted over 100 years and was finally lifted during the late 19th Century.
Russell Sage, an American financier and politician, is a notable figure in the history of options trading. He created call and put options that were traded over the counter in the US in the late 19th Century ("Russell," 2016). Sage has made millions by trading those newly created options that were highly unregulated and illiquid. There was a market crash in 1884 that could be perceived as the foreshadow to the Great Depression (Sobel, 1968). The market crash in 1884 was caused by the failure of Grant and Ward and Marine National Bank of New York (Sobel, 1968). The collapse of those two firms had caused many other firms to fail and created panic on the Wall Street. Russell Sage had lost his fortune and he quit trading but the option market continued to operate without any regulations. Although in that time, the formal exchange market was not established – there is belief that Sage was the first one who created a relationship between the price of the option, the price of its underlying security, and the interest rates.

Option trading was not progressing a lot due to the investors` skepticism. There have been many debacles caused by inconsistent option pricing so investors did not want to purchase or sell options. During the 1960s, the Chicago Board of Trade has noticed a significant decline in the trading at their exchange so they had to focus on a new approach that would make business grow. In fact, they realized that only the creation of the formal option exchange would promote and attract option trading. In 1973, the Chicago Board of Options and Exchange (CBOE) had begun trading option and that was the first time in the US that options contracts were properly standardized on the marketplace and made option trading more regulated.

In the beginning, there were only a few option contracts listed on the CBOE since many investors have remained averse towards option trading. The first listed contracts were mostly call options because many put options were not standardized at that time.
Two Professors, Fisher Black and Myron Scholes, came up with the mathematical formula that could calculate the price of an option using specific variables. Discovery of the Black Scholes Pricing Model had a positive impact on investors so they became more comfortable with option trading. In 1977, the put options were introduced on the CBOE and option trading started its expansion. According to Kiernan (2015), in 1982 more than 500,000 contracts were traded in a single day. Kiernan (2015) also writes that in 2014, 16.9 million contracts were traded on average per day. By comparing data from 1982 to 2014, it is possible to notice the expansion level in the options market. Graph 1 shows the expansion in equity option annual trade volume expressed in total number of contracts.

Graph 1: Equity Option Annual Trade Volume (1973 – 2007)

Chapter 2: Option Contracts

The main aim of this chapter is to define basic terminology and characteristics of option contracts that are widely used in finance. It also describes the classification of options based on exercise style and payoff calculation. This chapter also shows the general use of the options in investment and the role of the option pricing.

Option Terminology

An option is a common form of a derivative because an option derives its value from its underlying asset. In fact, most exchange-traded options have stocks as their underlying asset, but they could have any other type of security or commodity such as indexes, interest rates, bonds, currencies, swaps, baskets of assets or an economic goods such as real estate, water and electricity (McCafferty, 2011, p.3). Technically, it is possible to place an option on anything that can be purchased: coffee, cocoa, gold, silver, oil, gas, a plane, watch and so on.

People who buy options are called holders and people who sell options are called writers. Holding the option gives the buyer the right to do something with it, but it does not obligate the buyer (holder) to exercise the option. On the contrary, in forward and futures contracts the sellers and buyers have committed themselves to some action.

Unlike the stocks that investors can hold forever, the options have an expiration cycle defined in the binding contract. The buyer of the option has the right, but not the obligation to exercise the option at the expiration date. Know and Guthner (2016) write that “If the buyer`s choice is not to exercise the right of an option, it ceases to exist and the buyer and seller of the option go their own way” (p.12). Every option is linked to the binding contract, which strictly defines terms and properties related to that option.
Natenberg (2015) claims that “without the understanding of the terms of an option contract and the rights and responsibilities under that contract, a trader cannot hope to make the best use of options, not will be prepared for the very real risks of trading” (p.26).

Through the analysis of the contract specifications, all options are classified into two groups: call options and put options. A call option gives the buyer the right to purchase an asset at a certain price within a specific period. On the other hand, a put option gives the buyer the right to sell an asset at a certain price within a specific period. Simply, it is possible to notice that in option trading, all rights lie with the buyer and all obligations with the seller. A call option is a bullish instrument since the market participants expect that the price of the underlying entity (e.g. stock) will increase. On the contrary, a put option is a bearish instrument since the market participants expect the price decrease of the underlying asset.

There are four positions in option trading: Long Call, Long Put, Short Call, and Short Put. Option positions together with rights and obligations of buyers and sellers are summarized in Table 1. In fact, it is generally accepted that that option buyers have long position, and sellers have short position.

<table>
<thead>
<tr>
<th></th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>BUYER (Long)</td>
<td>Right to buy</td>
<td>Right to sell</td>
</tr>
<tr>
<td>SELLER (Short)</td>
<td>Obligation to sell</td>
<td>Obligation to buy</td>
</tr>
</tbody>
</table>

Source: *TD AMERITRADE, p.39*
From Table 1, it is possible to notice that while buying the call and/or put option does not carry any obligation, selling the call and/or put option obligates an investor to sell or buy a stock at the strike price, respectfully, if assigned by the option owner who exercised the option. Since holding options exposes investors to time decay, it is reasonable that investors want to have options with a longer expiration cycle, which could potentially enable the increase in the options’ value (Cohen, 2005, p.3). More particularly, Cohen (2005) explains that in respect to four option strategies, investors will buy calls and puts that will have at least three months left to expiration date (p.3). Furthermore, the investors will sell (short) calls and puts within a short time exercise cycle (preferably less than a month) which will bring investor short-term income.

The price of a particular option can be split into two components: the strike price and the intrinsic price. The strike price is fixed in the contract and specifies the price at which a specific derivative contract can be exercised. The intrinsic value is the difference between the underlying market price (spot price) and the strike price. For a call option, the option is “in-the-money” if the underlying price is higher than the strike price; then the intrinsic value is the underlying price minus the strike price. For a put option, the option is “in-the-money” if the strike price is higher than the underlying price; then the intrinsic value is the strike price minus the underlying price. It is possible to notice that if a call option is “in-the-money,” a put option with the same exercise price and underlying contract must be “out-of-money” and vice versa (Natenberg, 2015). Depending on the relationship between an option’s exercise price and the price of the underlying contract, an option could also be “at-the-money” and “out-of-the-money.” An option is “at-the-money” if the exercise price is the equal to the current price of the underlying contract.
An option is “out-of-money” if it does not have intrinsic value or simply the strike price exceeds the current spot price of the underlying asset.

Table 2: Call options

<table>
<thead>
<tr>
<th>In-the-money</th>
<th>Strike price &lt; Spot price</th>
</tr>
</thead>
<tbody>
<tr>
<td>At-the-money</td>
<td>Strike price = Spot price</td>
</tr>
<tr>
<td>Out-of-the-money</td>
<td>Strike price &gt; Spot price</td>
</tr>
</tbody>
</table>

When an exchange-traded option is exercised, depending on the terms specified in the contract, it can settle into three cases: 1) the physical underlying; 2) a futures position and 3) cash. The focus of this thesis will be the settlement into cash, which means that there will be a cash payment equal to the difference between the exercise price and the underlying price at the end of the trading day. Furthermore, it is important to point out the analysis of the seller behavior is very complex and requires a lot of advanced knowledge related to behavioral economics, this thesis will only focus on buyers of the options.

Each option is part of the particular style or family that is usually defined by the dates on which the option may be exercised. Styles can also depend on option complexity and the payoff calculation. The majority of options could be classified as European or American due to the fact that their payoff is calculated similarly. European and American options are known as vanilla options due to their level of complexity. On the other hand, there are exotic options that have specific characteristics, which can create challenges in the valuation and hedging process. Asian options, Bermudan options, Evergreen options, Russian options and Binary options are types of exotic options. From the exotic options
listed, it is possible to notice that the name of the option is mostly correlated to a particular geographic region. It is wrong to think, however, that the name of option is related to the options in the specific geographic region. For example, Asian options are the most basic form of exotic options and their main characteristic is that their payoff is determined by the average underlying price over some time. In 1987 Mark Standish and David Spaughton, workers in London-based Bankers Trust fixed derivative department, were in Tokyo on business when they developed the first commercially used pricing formula for options linked to the average price of crude oil (Palmer, 2010). They named this exotic option as the Asian option because they were in Asia when they created it.

The Options Clearing Corporation (OCC) is a vital entity in option trading. The main role of the OCC is to standardize option contracts, to guarantee the performance of option contracts, and to issue options (McCaffery, 2011, p.58). McCaffery also explains that standardization is the procedure that makes all option contracts interchangeable and liquid so they can be traded between investors (p.59). For example, it would be very hard to buy 25.75 shares of Apple and hope to find someone who would be willing to buy that identical contract. In the case of the issuance of a new option request by an exchange member, the exchange has to present the request and get the approval from the Security and Exchange Commission. When the new product is approved, OCC creates its specifications and settle options on the market.
Trading Options: Exchange Floor, OTC or Hybrid

Options trading can be floor-based, over-the-counter (OTC) or a hybrid of those two exchange types. The exchange type, however, should not have significant impact on the average trader at all. Although the exchange type should not necessary have an impact on the trader, it is crucial to observe the functioning and volatility of the exchange in general (McCafferty, p.58). It is possible to trade options on the following exchanges in the US: Chicago Board Options Exchange (CBOE), Chicago Mercantile Exchange (CME), Chicago Climate Exchange, Chicago Board of Trade and New York Mercantile Exchange (merged and acquired and now are CME Group), Kansas City Board of Trade (KCBT), Nadex, International Security Exchange (Eurex ISE), NASDAQ OMX, NASDAW OMX PHLX and many other (McCafferty, p.57).

Market Makers are considered important people in the process of the facilitating option trading. Their role is to quote a bid and offer (ask) price on the particular option. The bid represents the price at which Market Makers are willing to buy a particular option, and offer (ask) is the price at which Market Makers are willing to sell a particular option. The main goal of market makers is to increase the liquidity of the option market by ensuring the constant purchasing/selling of options at some quoted price without any delays (Hull, p.203).

It is important to mention that some exchanges have established the position limits and exercise limits. The position limit and exercise limit are usually equal and their main goal is to prevent the market from being excessively influenced by the activities of an investor or a group of them (Hull, p.203). The position limit refers to the maximum number of the option contracts that one investor can hold on the one side of the market (Hull, p.222). More precisely, long calls and short puts are on the same side of the
market, and short calls and long puts are on the same side of the market too. The exercise limit is defined as the maximum number of the option contracts that can be exercised by an individual, or the group of them acting together, in any period of five consecutive business days. There is a general belief that the position limits and exercise limits are not necessary for the proper function of the exchanges which makes them very controversial. The US exchanges are large and liquid so they cannot be easily influenced by an individual or the group of them.

Derivative markets are considered liquid, so they attract many different types of traders. In fact, traders can be categorized as into three categories: hedgers, speculators and arbitrageurs. Hedgers use options to reduce the risk that can be caused by the potential future movements in prices. Hedging can simply be compared to taking out an insurance policy that offers the protection from high price volatility. Speculators use options to bet on the future direction of a market volatility and they enable investors to limited loss to the amount paid for the option (Hull, p.10). Arbitrageurs take offsetting positions in two or more instruments to lock in a profit (Hull, p.10).

### Table 3: Hedging Strategies

<table>
<thead>
<tr>
<th>Option Position</th>
<th>Corresponding Market Position</th>
<th>Appropriate Hedge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy call(s)</td>
<td>Long</td>
<td>Sell underlying</td>
</tr>
<tr>
<td>Sell call(s)</td>
<td>Short</td>
<td>Buy underlying</td>
</tr>
<tr>
<td>Buy put(s)</td>
<td>Short</td>
<td>Buy underlying</td>
</tr>
<tr>
<td>Sell put(s)</td>
<td>Long</td>
<td>Sell underlying</td>
</tr>
</tbody>
</table>

Source: Natenberg, 2015, p.64
“For new traders, it may be helpful to point out that we are always doing the opposite with calls and underlying (i.e., buy calls, sell the underlying; sell calls, buy the underlying) and doing the same with puts and the underlying (i.e., buy puts, buy the underlying; sell puts, sell the underlying),” said Natenberg (p.65). In addition, Natenberg points out that the strategies involving the sale or purchase of single option are dominant strategies used by majority of hedgers. It is important to keep in mind that Natenberg’s rational is that the majority of hedgers are not professional option traders and do not have the time and desire to develop more complex hedging strategies.

This chapter described the fundamental concepts that are necessary to understand option trading. It gave a good overview of the option classification and explained call options and put options. This chapter also distinguished between European and American options. The next chapter is going to examine the valuation techniques that are commonly used to value European and American call options.
Chapter 3: Methods for Options Pricing

Because the value of an option contract does not depend only on the value of the underlying asset, but also on many other variables such as market volatility and option exercise time, the option pricing procedure is more complex. The aim of this chapter is to investigate mathematical models that are used for pricing options. Many different valuation models have been developed, but not all of them can incorporate all variables that have impact on the option price. Commonly used models in options pricing are: 1) The Black-Scholes-Merton Option pricing; 2) The Monte Carlo Simulation; and 3) Lattice models: Binomial and Trinomial tree. Although there are many different valuation models used in option pricing, this thesis will only focus on two frequently used models in the literature: The Black-Scholes-Merton Model and The Monte Carlo Method.

Black-Scholes-Merton Option Pricing Model

The Black-Scholes Option Pricing Model had a crucial role in development of the option markets because it was the first mathematical model established with the purpose to valuate options. It is used for the development of many other pricing models. In 1970 Fischer Black, a mathematical physicist, and Myron Scholes, a professor of finance at Stanford University, wrote a paper titled *The Pricing of Options and Corporate Liabilities*. The authors tried to publish the paper, but various publishers rejected it. They managed to publish the paper in 1973 in Chicago University’s *Journal of Political Economy*. Black and Scholes claimed that every option has a correct price, which could be determined by using their equation known as the Black-Scholes formula. In 1973, a subsequent paper, *Theory of Rational Option Pricing*, was written by Robert Merton with the aim to expand on the Black-Scholes option-pricing model. Black, Scholes and Merton
have shown the application of differential equations to determine a fair value of European style calls and puts. Merton and Scholes were awarded the Nobel Prize in Economics in 1997 in honor of development of the model – two years after the death of Fischer Black.

The assumptions of the model are:

1. Options can be exercised only at expiration.
2. Risk-free rate and volatility are constant over the life of the option.
3. The underlying security does not pay dividends.
4. The underlying security will sometimes go up in price and sometimes go down and that the direction of the movement cannot be predicted.
5. There is no commission charged on the purchase or the sale of the option.
6. There is no arbitrage opportunity.

Natenberg (2015) claims that “Black-Scholes model, with its very simple arithmetic and limited number of inputs, most of which were easily observable, proved an ideal tool for traders in the newly opened U. S. option market” (p.62 ). In fact, in order to calculate an option’s theoretical value by using the Black-Scholes model, we need to know at least five identifiers of the option and its underlying contract. The five crucial variables of the Black-Scholes model are:

1. The option’s exercise price (strike price): the price is fixed and it is defined in the binding contract.
2. The time remaining to expiration: it is fixed and cannot vary.
3. The current price of the underlying contract: the correct price of the underlying contract is not always obvious. There is a bid price and ask price (bid-ask spread) and it may not be clear whether a trader should use the bid price, ask price or some value in between. The current price of the underlying
contract should be the price for which the trader believes that she/he can make the most favorable trade.

4. The applicable interest rate over the life of the option: due to the fact that the option trader may end up with the cash credit or debit on his/her trading account –the interest generated by those cash flows should be considered. Usually, it is suggested to use the prime risk-free rate (the rate that applies to the most secure borrowers).

5. The volatility of the underlying contract – this is the most difficult variable to be understood. Volatility could be considered as the most important variable in actual trading. Any change of assumptions related to the volatility could have significant effect on the option`s value.

**Diagram 1: Schematic of Black Scholes Model**

![Diagram of Black Scholes Model](source.png)

Source: Natanberg, p.63
**Derivation of Black-Scholes Model**

Equation (1.1) is used to describe the stock prices behavior in the real world. The variable \( \mu \) represents the stock`s expected rate of return. Furthermore, the variable \( \sigma \) measures the volatility of the stock`s price. In case of risk-neutral valuation, the variable \( \sigma \) should be replaced with the current risk-free rate. The equation 1.1 is known as generalized Weiner process.

\[
dS = \mu S \, dt + \sigma S \, dz \tag{1.1}
\]

Let suppose that we have an option whose value is determined only by \( S \) and \( t \). At this point, let just say that the same formula will be used for call options, put options or portfolios. By using Ito`s lemma, a mathematical identity used in stochastic calculus is generated:

\[
df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S \, dz \tag{1.2}
\]

Equation (1.3) is a discrete form of equation (1.1)

\[
\Delta S = \mu S \Delta t + \sigma S \Delta z \tag{1.3}
\]

Equation (1.4) is a discrete form of equation (1.2)

\[
\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z \tag{1.4}
\]

It is possible to construct the portfolio of the stock and the derivative so the Wiener process will be eliminated. The value of portfolio is expressed through the equation 1.5 below.

\[
\Pi = \cdot f + \frac{\partial f}{\partial S} S \tag{1.5}
\]
Equation (1.6) shows the rate of change in portfolio value cause by the change in some time interval.

$$\Delta \Pi = - \Delta f + \frac{\partial f}{\partial S} \Delta S$$ \hspace{1cm} (1.6)

By substituting equations (1.3) and (1.4) into (1.6) the following is created:

$$\Delta \Pi = (\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2) \Delta t$$ \hspace{1cm} (1.7)

In fact, the risk-neutral form of equation (1.7) will be expressed as (1.8) where \( r \) is the risk-free interest rate.

$$\Delta \Pi = r \Pi \Delta t$$ \hspace{1cm} (1.8)

By substituting equations (1.5) and (1.7) into equation (1.8) will create the famous Black-Scholes-Merton differential equation

$$\left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \right) \Delta t = r \left( f - \frac{\partial f}{\partial S} S \right) \Delta t$$ \hspace{1cm} (1.9)

which can be rewritten as

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$ \hspace{1cm} (1.10)

The Black-Scholes-Merton differential equation can be solved as (1.11) for call options.

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$ \hspace{1cm} (1.11)

Where \( S_0 \) is current stock price, \( K \) is option strike price, \( T \) is time until option expiration, \( r \) is the risk-free rate, \( N \) is cumulative normal distribution, \( e = 2.7183 \). Furthermore,

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$ \hspace{1cm} and \hspace{1cm} \( d_2 = d_1 - s \sqrt{T} \). (Where \( s \) is the standard deviation of stock returns). By inserting parameters into equation (1.11) it is easy and fast to estimate theoretical value of European call options.
Limitation of the Black-Scholes Model

It is crucial to keep in mind that this model assumes that the risk-free rate and volatility are assumed to be constant over the option’s life-span. This assumption is very unlikely to be encountered in option trading. Furthermore, the other major assumption is that the option can only be exercised at the maturity, but not all options have the same maturity date. In fact, this model loses its magic when dealing with vanilla options: American options, since they can be exercised anytime during their lifetime. It is possible to valuate European option using the Black-Scholes-Merton model because European options can be exercised only at the end of their lives. In contrast, the Black-Sholes-Merton model fails in the process of valuation American options because American options have a more flexible exercise rule and can be exercised at any time before the expiration date.

The Monte Carlo Simulation

The Monte Carlo Simulation is one of the most important and powerful algorithms in finance. Its application is very common in option pricing, risk management and financial modeling. The advantage of this algorithm is the ability to handle complex and high-dimension problems. There are many different types of the Monte Carlo Algorithm, but the General Monte Carlo and Least-Squares Monte Carlo will be examined and used to estimate the theoretical value of European and American call options.
History of the Monte Carlo Simulation

David B. Hertz was the first one who suggested using the Monte Carlo Simulation in business applications in 1964 (Nowrocki, 2001). The general application of the Monte Carlo Simulation had application in different fields before 1964. Specifically, “the idea of using randomness in a determinative manner” can be traced back at least to the 18th Century (Harrison, 2010). The use of the Monte Carlo Simulation was present in studies done by different scientists such as Georges Louis LeClerc-Comte de Buffon (1707-1788), an influential French scientist, who used random methods in a number of studies. His most famous study is “Buffon’s needle” where he presented a method using repeated needle tosses onto a linked background to estimate pi-value. Other examples of the early application of the Monte Carlo Method is G. H. Darwin (1845-1912; Charles Darwin's son) who used algorithms to smooth curves. In similar fashion, F. Galton (1822-1911; C. Darwin's nephew) has described a general simulation method using modified dice in order to generate a normal distribution.

Harrison (2010) states that there is a significant difference between those seminal studies and typical modern Monte Carlo simulations. The author claims, “The early simulations dealt with deterministic problems; modern simulation inverts the process, treating deterministic problems by first finding previously understood a probabilistic analog and solving the problem probabilistically” (p.2).

The fundamental methods of the first modern Monte Carlo Simulation were established and used by John von Neumann and Stanislaw Ulam. In fact, Stan Ulam came up with the idea while he was recovering from illness and playing solitaire (card game). Ulam tried to calculate the likelihood of winning based on the initial layout of cards using combinatorial calculations (Eckhardt, 1987). Due to the calculation complexity, Ulam
decided to focus more on a practical approach by trying many different layouts and observing the number of successful games (Eckhardt).

Ulam and Van Neumann had suggested the use of their simulation to investigate the properties of neutron traveling through radiation shielding. This simulation was part of the secret Manhattan Project during World War II when the US government had plans to develop nuclear weapons. Due to the fact that the chance and random outcomes are central to the Monte Carlo method, Ulm and Van Neumann have named The Monte Carlo Method in reference to popular gambling center Monte Carlo.

**General Monte Carlo Method**

A Monte Carlo simulation can be simply explained as a mathematical model that performs analysis by imputing a range of values (probability distribution) in order to offer distributions of possible outcome values. Probability distributions used in the Monte Carlo may differ but Normal, Lognormal, Uniform, Triangular, and Discrete are the most commonly used. “The Monte Carlo simulation is a specialized probability application that is no more than an equation where the variables have been replaced with a random number generator” (Srivastava & Bajaj, 2014, p.60).\(^1\) This simulation could involve hundreds of thousands of recalculations before it can be completed. Through the repetition of the Monte Carlo Method, the precision improves due to the Law of Large Numbers. Specifically, by performing the experiment \(n\) times, each time in the same conditions and each time independently of each other, the average value is very close to the expected value with very high probability.
Since the Monte Carlo Simulation has gained popularity in many different disciplines ranging from the natural sciences to finance, its diverse application has caused adjustments of the method and the creation of many different versions of the Monte Carlo simulation. Hull (2012, p.470) summarize the five fundamental steps of The Monte Carlo Simulation used to estimate the value of particular derivative. The author assumes that the interest rate is constant in the economy.

**Five-Step Monte Carlo Method:**

1. To sample a random path for S in a risk-neutral world

2. To calculate derivative payoff using formula $S_t = S_0 \exp((r - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T}z)$

3. To keep repeating step 1 and step 2 in order to collect many sample values of the payoff from derivative in as risk-neutral environment

4. To calculate the average of the sample payoffs in order to get an estimate of the expected payoffs in a risk-neutral world

5. Discount the expected payoff at the risk-free rate using The Monte Carlo Estimator: $C_0 \approx e^{-rT} \frac{1}{I} \sum_{i} payoff$

The Monte Carlo Simulation has gained popularity in many different disciplines ranging from the natural sciences to finance. Diverse application of MCS lead to adjustment of the original version and creation of many other versions (e.g. Least Squares Monte Carlo). The Least Squares Monte Carlo is based on the Longstaff-Schwartz model from 2001. This model is based on the use of an ordinary least-squares regression to estimate the conditional expected payoff to the investor from continuation (Longstaff and Schwartz, 2001, p.113). Hull (2012) writes that the main role of LSM is to “determine
the best-fit relationship between the value of continuing and the values of relevant variables at each time an early exercise decision has to be made” (p.646).

**Advantages and Disadvantage of The Monte Carlo Simulation**

The Monte Carlo Simulation is more efficient than other models when there are more than three stochastic variables included in the calculation. In fact, the time needed to perform The Monte Carlo Simulation increases approximately linearly with the number of variables (other models usually have exponential time complexity). The Monte Carlo Simulation requires a lot of memory even for the simplest programs which implies that efficient implementation of it is crucial.
Chapter 4: Literature Summary

The main aim of this chapter is to explore the techniques that have been commonly used in option valuation. In fact, there is a considerable amount of literature on the use of The Black-Scholes Model and The Monte Carlo Simulation in option pricing. This chapter also includes some studies that show the use of The Monte Carlo Simulation in other branches of finance such as real estate and risk management.


The second half of the 20th Century until the financial and economic crisis in 2007 was marked by a very dynamic development in financial markets and banking (Balling and Gnan, 2013, p.157). Balling and Gnan argue that “there was a development trend towards liberalization and the global integration of financial markets and banks which lead to the increase in sophistication, complexity and inter-connectives on the financial market after the World Was Two” (p.157). They explain that development in financial markets and banking would not have been possible without the intellectual underpinning of economic arguments for free markets, international division of labor, globalization and without advances in financial theory and the related improvements in statistical estimation methods used to price new products. The authors assert that financial markets evolved gradually with some segments such as derivatives markets being non-existent 50 years ago which implies that some financial market data are not available at all.

Sundaresan (2009) states that the governments and central banks may have a vested interest in developing liquid financial markets for the conduct of their fiscal and monetary policy implementation. In order to explain the central bank interest in development of the financial markets, Sundaresan uses the example of extensive over-the-counter (OTC) dealer network that allows the central bank to implement its monetary
policy by increasing or decreasing the money supply. Sundaresan (2009) draws the conclusion that financial markets in the Western economies (defined to include United States, the United Kingdom and Western Europe) have developed rapidly mostly as the consequence of the market crisis and action of the regulatory bodies. A financial crisis makes financial market actors more focused on the actions that should be taken in order to make financial markets more resilient. The author claims that investors and issuers in the Western economies use these markets to apply their risk-return assessments in their investment and issuance strategies.

Financial markets play an important role in economic development by facilitating the collection of savings and by channeling funds to investors (Wai, 1976). More precisely, Wai perceives that since economic growth depends largely on investment – the financial markets can play an important part in: a) channeling savings into investment, b) increasing the quantity of investment, and c) increasing the quality of investment.

Similarly, Adam (2009) through his paper Financial Markets: The Recent Experience of a Developing Economy examines the evaluation of Sudanese financial markets in general and infant capital markets in particular in the period 1995-2004. The author’s main idea is that financial markets, institutions and instruments that build the financial sector, play a crucial role in the financial development and economic growth of a developing economy.
**Options: European, American and Asian**

The first systematic study on pricing American options by using the Monte Carlo method was conducted in 1977 by Phelim Boyle. In his groundbreaking paper of 1977, Boyle uses a simulation method to obtain numerical estimates of a European call option on a stock which pays discrete dividends. The additional assumption that has been presented through Boyle’s paper is that the returns on the underlying stock follows a lognormal distribution. Other studies, for Black and Scholes (1973), Cox and Ross (1976), Broadie and Glasserman (1995), and Longstaff and Swartz (2001) have been conducted on the use of the Monte Carlo Method in options pricing and they provide useful insight in process of option valuation.

Many problems in finance and financial engineering focus on estimating a certain value of different financial instruments, portfolios and investment strategies (Chen & Hong, 2007). Chen and Hong claim that the Monte Carlo simulation is a method that is often used to estimate those expectations. In addition to Chen & Hong, Glasserman (2003) concludes that “Monte Carlo simulation has become an essential tool in the pricing of derivative securities and in risk management; these applications have, in turn, stimulated research into new Monte Carlo techniques and renewed interest in some old techniques” (p.1).

Rubinstein (1981) develops a set of criteria to be used in deciding whether it is appropriate to use a Monte Carlo simulation. According to Rubinstein, Monte Carlo simulation is appropriate only when: 1) it is impossible or too expensive to obtain data; 2) the observed system is too complex; 3) the analytical solution is difficult to obtain; and 4) it is impossible or too costly to validate the mathematical experiment. The criteria written above is also confirmed through the study done by Rees and Sutcliffe (1993) and
Nowrocki (2010) whose main idea is that “Monte Carlo simulation is useful only when nothing else will work” (p.2).

Recent evidence suggests that the Monte Carlo simulation is the most powerful numerical technique for the valuation of American options (Miao and Lee, 2012). Miao and Lee point out that the Monte Carlo method has higher flexibility, wider applicability to various products and lower sensibility to the problem dimension which makes this simulation to be better suited to pricing problems that are include multiple assets. Although Miao and Lee argue that the Monte Carlo simulation is the most powerful numerical technique for valuation of American options, it is important to add that they assert that the Monte Carlo method generally performs worse in American than in European options pricing problems due to its free boundary nature (ability of early exercise).

The main idea of Miao`s and Lee`s paper is to address the issue of inefficiency in option pricing by proposing a new Monte Carlo method that requires only forward evolution. Miao and Lee have omitted the backward induction in the Monte Carlo method in order to avoid the need for future information about option prices. Their Forward Method has to determine if it is optimal to exercise based on the current stock price. In order to determine if the stock entered the exercise boundary, the authors used specifications of an early exercise boundary proposed by Barone-Adesi and Whaley (1987).

Due to the fact that the Monte Carlo algorithm in Miao`s and Lee`s research does not use backward induction it provides its users with several advantages. First, the algorithm has faster execution which requires less time. Second, it does not store all simulated paths which makes it more efficient in terms of space (memory) required.
Whenever the stock price moves a step forward, the past stock price is not needed and is discarded.

Lin (2008) similarly investigated three different pricing algorithms of American options through the implementation of the Monte Carlo simulation. In fact, Lin has implemented those three algorithms to price American vanilla puts and to get numerical results for further analysis. Lin uses two low-biased algorithms and one high biased algorithm in order to conduct his research. Lin claims that the low-biased algorithms are the Longstaff-Schwartz algorithm and an alternative low-biased algorithm based on a sub-optimal exercising policy. The high-biased algorithm originates from dual formulations and its main feature is the use of sub-paths in approximating a certain type of conditional expectations required for the valuation. Lin (2008) reveals that the use of simulation in American option pricing is advantageous because the simulation intends to solve discretized dynamically programming problem. This thesis offers a good overview of pricing methods for the simple case of an American-style option, but it would offer deeper understanding of material by testing those algorithms on more complex American options.

Cvetanovska and Stojanovski (2012) emphasize the importance of an American option in computational finance and the challenges that they have encountered during implementation of the Monte Carlo methods. They agree that American options represent a challenging problem in computational finance due to their early exercise feature and the inability to approximate in continuous time. Cvetanovska and Stojanovski point out that the use of the Quasi Monte Carlo simulation gives the best convergence and most accurate price for this type of option. The difference between the Monte Carlo method and the Quasi Monte Carlo is that the points in Monte Carlo are randomly chosen and
independent – unlike Quasi Monte Carlo where the points are generated quasi randomly (sub-randomly). Cvetanovska and Stojanovski have based their valuation on a single option with a large number of stock pricing paths needed to ensure higher accuracy of the Monte Carlo simulation. In order to conduct their experiment, Cvetanovska and Stojanovski performed permutation on quasi-random arrays by using linear congruential generator that produces statistically independent samples for each path. Each of those samples is normally distributed $N(0, 1)$ by using Moro Inverse Cumulative Normal Distribution. In contrast to Miao and Lee’s algorithm that uses forward evaluation, their algorithm is implemented based on backward induction of a dynamic programing principle. Cvetanovska and Stojanovski assume that all future values of underlying stock prices are known so it is possible to choose the best point in time to exercise the option. Their algorithm is capable of simulating more than one million paths for a Monte Carlo simulation in approximately 0.15 microseconds. They conclude by stating that pricing of an American option is an active area of academic research that is attractive and relatively new in the field of finance.

Zhang (2009) research is focused on the pricing of an arithmetic average Asian option with the help of the Monte Carlo method. Since Asian options are classified as derivatives, the value of the Asian option is derived from the other underlying asset (e.g. stocks, an index portfolio). Zhang defines Asian options as options in which the underlying variable is the average price over a period of time. Furthermore, Zhang offers the empirical proof that Asian options have a lower volatility which makes them cheaper relative to their European counterparties. According to Zhang, the lower volatility of the Asian option could offer the protection against the price manipulation. In order to explain the concept of protection against price manipulation, Zhang draws the following
theoretical example. He states that, “if a standard European call option is based on a stock which remains low in price during a large part of the final time period and rises significantly at maturity, the option writer would have to face a massive loss; whereas the average option could avoid such manipulation” (p.13). Zhang concludes that the use of the Monte Carlo method for pricing Asian options, and other path dependent option, is a challenge due to the large number of calculations that are needed to update the path dependent variable throughout the simulation. Similarly to Stojanovski and Cvetanovska (2012), Zhang proposes the use of Quasi Monte Carlo methods as a potential improvement in pricing Asian options.

Glasserman (2003) is focused on understanding the importance of the source of the high and low bias that affect all methods for pricing American options by using simulations. Glasserman defines high bias as a result from using information about the future in making the decision to exercise which leads to the application of a backward induction to simulated paths. On the contrary, low bias results from following a suboptimal exercise rule. In the end, Glasserman concludes that some methods mix high and low bias but by separating them often leads to production of a pair of estimates straddling the optimal value.

**Bonds and Hybrid Financial Instruments**

The Monte Carlo Simulation is also an useful tool in the valuation of bonds, securities and other hybrid financial instruments. In their paper of 2004, Lvov and Yigitbasioglu have proposed a pricing methodology for convertible bonds that is based on the application of the Least-squares Monte Carlo methodology. Throughout their paper, Lvov and Yigitbasioglu identify the convertible bonds as complex hybrid securities which are subject to multiple sources of risk (e.g. equity risk, interest rate risk,
volatility and credit risk). The Least-squares Monte Carlo methodology was first introduced in the literature by Longstaff and Schwartz in 2001 with the main purpose to value American options. Lvo and Yigitbasioglu claim that their paper “breaks away” from the tradition established in the literature of pricing convertible bonds with finite difference and lattice methods through the use of Monte Carlo pricing simulation. The authors have compared the result of Least-square Monte Carlo simulation for convertible bonds against an accurate Finite Difference scheme, and the result showed that the methodology is reliable. Lvo’s and Yigitbasioglu’s algorithm performs well, but still need improvement in accuracy and speed. The authors also propose the use of additional risk factors such as default intensity in order to test the simulation.

Ammann et al. (2008) use an empirical study of the pricing model for convertible bonds based on the Monte Carlo Simulation. Since a convertible bond is a hybrid financial instrument, it means that a convertible bond has a complex structure that makes its valuation more difficult by using only analytical and numerical analysis. The authors claim that “Convertible bonds depend on variables related to the underlying stock (price dynamics), fixed income part (interest rates and credit risk) and the interaction between those two components” (p.2). Ammann and co-authors claim that use of the Monte Carlo simulation will allow them to better capture both the dynamics of the underlying state variables and the rich set of real-world convertible bond specifications. Their empirical study is based on examining the US market through the use of 32 convertible bonds and 69 months of daily market prices obtained from Mace Advisors. The value of the convertible bond, given certain exercise strategy, is calculated by averaging the discounted payoffs of all simulation paths. They concluded that theoretical price values for the analyzed convertible bonds are on average 0.36% lower than observed market
prices, with a RMSE of 6.18% (p.25).\(^5\) While previous studies claimed that there is average overvaluation (model prices are higher than market prices), their study showed opposite results.

Ding et al (2012) use the Monte Carlo Simulation technique in order to price callable bonds.\(^6\) Ding and co-authors analyze Monte Carlo Simulation for the callable bonds with several call dates under the Cox-Ingersoll-Ross (CIR) interest rate model (p.121). Authors have drawn the parallels between bonds (derivatives) and options by saying that a callable bond is a straight bond embedded with a call of the European option (a single call date) or Bermudan option (several call dates). It is important to emphasize that Ding et al offer the numerical experiments for a practical callable bond issued by the Swiss Confederation with ten call dates and two months’ notice. In order to determine the efficiency of the Monte Carlo Simulation, Ding and co-authors have performed a numerical experiment on the same callable bond using three different techniques: the balanced implicit method (BIM), the balanced Milstein method (BMM) and the exact transition distribution method (ETD).\(^7\) Through an analysis of the results, it is possible to see that the Monte Carlo Method and the BMM method are more efficient than the other two methods. This implies that the Monte Carlo Method works well for pricing callable bonds.

**Other Application of Monte Carlo Simulation in Finance**

Although the main focus of my thesis is the application of the Monte Carlo Simulation in financial instruments valuation, showing the wide application of the Monte Carlo Simulation in other areas of finance establishes the simulation’s power. For example, Kwak and Ingall (2007) define the Monte Carlo simulation as a useful technique for modeling and analyzing real-world systems and situations. Throughout the
conceptual paper *Exploring Monte Carlo Simulation Applications for Project Management*, the authors explore the applications of Monte Carlo Simulation and its relevance to managing project risk and uncertainties. Furthermore, Kwak and Ingall conclude by pointing out that Monte Carlo is simple and useful tool and that more project managers should take advantage of it.

Jayarman (2013) writes about the application of Monte Carlo Method in Real Estate Finance. The author claims that estimating the value of real estate property is important to a variety of endeavors including real estate financing, real estate listing, investment analysis, property insurance and the taxation of real estate (p.2). Furthermore, Jayarman points out that results of Monte Carlo methods are not single value estimates but a stream of possible values moving from the worst to the best estimate, which allows a user to make an informed decision regarding the investment as the risk involved is factored into the model (p.4). Jayarman’s paper is just a theoretical approach and it would be more desirable to support her claim by performing analysis on real estate assets with the Monte Carlo Simulation.

**Conclusion**

From the literature review, it is clear that the use of the Monte Carlo Simulation in pricing options is researched by many different authors. There is consensus that the Monte Carlo Simulation is the appropriate valuation technique for European, American and Asian options. Similarly, it is possible to see that the Monte Carlo Simulation offers efficient and accurate valuation of collaterals and callable. There is lack of academic research done on the individual stock valuation using the Monte Carlo Simulation. This is a vital issue for further empirical research.
Chapter 5: Empirical Analysis

The main aim of this chapter is to compare the results obtained from different call option pricing models. The properties of Google Option Contract, SPX Option Contract and Apple Option Contract have been used to facilitate the valuation process. Three different valuation cases are examined and their results are presented. This thesis hypothesizes that the Monte Carlo Method (general) is the most efficient valuation tool due to its flexibility.

Due to the fact that the main goal of every trader is to make a profit, it is clear that traders want to buy options below their fair value and to sell options when they exceed their fair value. As previously discussed, the valuation of option prices is an active area of research in academia and the financial industry, and there are many researchers who are trying to discover the option valuation tool that will price options accurately. Since options are complex financial instruments, it is essential to observe all factors that have an impact on option prices. Those factors are the following: 1) the current stock price (So); 2) the strike price (K); 3) the time to expiration (T); 4) the volatility of the stock price (\( \sigma \)); 5) the risk-free interest rate (\( r \)); and 6) the dividends that are expected to be paid. The predicted effect on the value of an option caused by an increase in one of these six variables while keeping the rest fixed is summarized in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>European call</th>
<th>European put</th>
<th>American call</th>
<th>American put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current stock price</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Strike price</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Time to expiration</td>
<td>?</td>
<td>?</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Volatility</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Amount of future dividends</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

+ indicates that an increase in the variable causes the option price to increase or stay the same; - indicates that an increase in the variable causes the option price to decrease or stay the same; ? indicates that the relationship is uncertain.

Source: Hull, 2012, p.235
When a call options is exercised, the payoff is equal to the difference between the current underlying stock price and the strike price. The increase in the current underlying stock price, while all other variables are held constant, will lead to increase in the value of call options. On the other hand, an increase in the strike price, while all other variables are held constant, will bring the option value down due to the fact that net payoff will decrease.

Volatility can be perceived as the most challenging variable that is involved in option valuation. Hull (2012) writes that “the volatility of a stock price is a measure how uncertain we are about the future stock price movements” (p.235). Investors that held call options profit from price increases. In the case of a price decrease, investors that hold call options experience a limited loss because the most that they can lose is the premium paid to purchase option rights since they won’t exercise the option in this case. There is the Risk-Return Tradeoff between the increase in volatility and the option value. The increase in the volatility, will lead to a potential increase in the value of call options.

Dividends have the opposite effect on the call and put options. In particular, the existence of future dividends will lead to a decrease in the underlying stock price of a call option. The current underlying stock price of a call option will be decreased by the amount of dividend paid. A decrease in the underlying stock price, while all other variables are fixed, will lead to a decrease in the value of call options.

An increase in the Risk-free interest rate, keeping everything else fixed, will increase the call option value. Hull (2012) writes that “when interest rates in the economy increase, the expected return required by investors from the stock tends to increase” (p.237). Furthermore, the author states that an increase in interest rates will decrease the present value of the future cash flows that option holders receive (p.237).
Those two effects are combined and they together have a positive impact on the value of call options.

American call options become more valuable with the increase in time to expiration. The longer time horizon means that there is more time for some positive or negative event to occur and have an impact on the option value. In case of European call options, the impact of time to expiration is undetermined. Hull (2012) gives a great example that illustrates this phenomena. He assumes the existence of two options: Option A which expires in one month and Option B which expires in two months. Furthermore, Hull assumes that there is an expected large dividend in the next six weeks. The expected dividend will impact option B – there will be a decrease in the underlying stock price and value of Option B. In this case, the one-month Option A will be more valuable than the two-month Option B.

**Research Aim and Hypothesis**

The main goal of this thesis is to investigate the analytical and mathematical methods used in estimation of the fair market value of the European and American call options. All algorithms are created in Python which is a free and powerful programming language with a lot of packages that are widely used in finance (see Appendix). In fact, Python is commonly used in investment banking, risk management, analyzing data and building different models. There are many libraries such as Numpy, Scipy and pandas that are very useful when dealing with scientific computing. It is interesting to point out that majority of options belong to the American exercise style. Options on stock indices and commodities, however, are mostly European exercise style. In order to perform the empirical work, a Google Option Contract, Apple Option Contract and SPX Option
Contract parameters are used. Google and Apple option contracts both belong to the American exercise style. The Google option contract does not pay any dividend in contrast to the Apple option contract. This thesis examines two option contracts that are in the same sector (technology) and have different characteristics. The SPX Option Contract generally mirrors the performance of 500 leading US firm and it is very popular investment tool for risk averse investors.

**Case 1: Empirical Work, Methodology and Results for American Call Option with no Dividends (GOOGLE)**

An American call option that does not pay dividends is considered to be the most simple type of American options due to the fact it requires only a simple payoff calculation. Although American call options can be exercised at any time during their life time, it is not optimal to exercise them until the options’ maturity because the value of American call option increases with the increase in time horizon (recall Table 1). In fact, the valuation of American call options that do not pay dividends should be identical to the valuation of European call options due to the fixed exercise time recommended. So, an European call option can be valued using the same pricing models for an American call option without dividends. The following valuation models have been used in order to price American call options that do not pay dividends: 1) The Black Scholes Model; 2) The Monte Carlo Method; and 3) A Method that is a combination of the-Black-Scholes Model and the Monte Carlo Method.

The empirical work is based on input of Google Option Contract parameters in algorithms built based on The Black Scholes Model and The Monte Carlo Method (refer
to Appendix). The results of simulation prices of an American call with no dividends are presented in Table 2. When an option is in-the-money (stock price > strike price), all three methods produce approximately the same results of option fair market value. In fact, the difference between the results of three methods (BSM = 237.36, MC = 237.42, Combined = 237.34) presented below when simulations are iterated 400,000 times does not exceed 1%. So, it is possible to conclude that the Monte Carlo Method and Combined Method benefit greatly with the increase in iteration. Furthermore, if we observe The Monte Carlo Method and Combined Method when Google Option Contract is at-the-money, it is possible to notice that at the low iteration level (I = 100) that the Combined Method (237.43) performs better in comparison to the General Monte Carlo Method (235.30). In fact, the difference between The Black Scholes Model (237.36) and Combined Model (237.43) is only 0.03%.

**Table 2: Simulation Prices American Call in-the-money with no dividends (GOOGLE)**

<table>
<thead>
<tr>
<th></th>
<th>BSM Analytical</th>
<th>General Monte Carlo</th>
<th>Combined</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>237.36</strong></td>
<td>235.30 (I = 100)</td>
<td>237.01 (I = 10,000)</td>
<td>237.43 (I = 100)</td>
</tr>
<tr>
<td></td>
<td>237.33 (I = 100,000)</td>
<td>237.40 (I = 100,000)</td>
<td>237.49 (I = 10,000)</td>
</tr>
<tr>
<td></td>
<td>237.42 (I = 200,000)</td>
<td>237.41 (I = 100,000)</td>
<td>237.38 (I = 200,000)</td>
</tr>
<tr>
<td></td>
<td>237.42 (I = 400,000)</td>
<td>237.38 (I = 100,000)</td>
<td>237.43 (I = 400,000)</td>
</tr>
</tbody>
</table>

Furthermore, from **Chart 1** it is possible to see that with the increase of the number of iterations, The Monte Carlo Method and The Combined Method converge. The reason for this convergence is the Law of Large numbers- as the number of iterations increases, the actual result will converge to its theoretical value.
Table 3 is used to make qualitative statements and better visual comparison of fair market values of the Google option contract produced by alternative formulas (General Monte Carlo Method and Combined Method) relative to the values of those same options according to the Black Scholes Model.

Let consider the market value when the Google option contract is at-the-money (Stock price = Strike price). When expiration time is five days, the market value is estimated to be the following: The Black Scholes Model (12.90), The General Monte Carlo (12.92) and Combined Method (12.94). The greatest difference is between The Black Scholes Model and Combined Method – 0.31 %. When expiration time is 252 trading days, at-the-money Google contract’s market value is estimated as following: BSM (96.33), MC (96.50) and Combined (96.51). Again, the greatest difference between two models is between The Black Scholes Model and Combine Model, but it is insignificant (0.19 %). So, it is possible to conclude that for at-the-money American call
options that do not pay dividends – all three methods are equally good predictors of the option value.

When the Google option contract is out-of-money, for the short expiration time (five and ten days) all three models predict that the market value will be equal to zero. With the increase in time to expiration (126 and 252 days), the low market value is estimated and it is approximately the same for all three models. Out-of-money option is going to expire being worthless and the investor will lose his initial premium paid to purchase the call option rights.

**Guide: Table 3** provides answers to following two questions:

1. If fixed strike price is given, how will time to expiration impact the market value of call option?

2. If fixed time to expiration is given, how will strike price impact the market value of call option?
Table 3: Comparative Option Values GOOGLE

\[ S_0 = 837.17 \text{, } r=1.56\% \text{, } \sigma = 4.67 \% \text{, Iterations = 400,000} \]

Time to Expiration (in days)

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>5</th>
<th>10</th>
<th>126</th>
<th>252</th>
<th>5</th>
<th>10</th>
<th>126</th>
<th>252</th>
<th>5</th>
<th>10</th>
<th>126</th>
<th>252</th>
</tr>
</thead>
<tbody>
<tr>
<td>BSM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>737.20</td>
<td>737.23</td>
<td>737.95</td>
<td>738.72</td>
<td>737.26</td>
<td>737.32</td>
<td>738.25</td>
<td>739.14</td>
<td>737.22</td>
<td>737.26</td>
<td>738.00</td>
<td>738.56</td>
</tr>
<tr>
<td>500</td>
<td>337.32</td>
<td>337.48</td>
<td>341.18</td>
<td>346.56</td>
<td>337.39</td>
<td>337.57</td>
<td>341.48</td>
<td>346.95</td>
<td>337.32</td>
<td>337.43</td>
<td>341.08</td>
<td>346.73</td>
</tr>
<tr>
<td>600</td>
<td>237.36</td>
<td>237.54</td>
<td>243.89</td>
<td>255.39</td>
<td>237.42</td>
<td>237.63</td>
<td>244.16</td>
<td>255.76</td>
<td>237.40</td>
<td>237.47</td>
<td>243.49</td>
<td>255.62</td>
</tr>
<tr>
<td>837.17</td>
<td>12.90</td>
<td>18.32</td>
<td>67.09</td>
<td>96.33</td>
<td>12.92</td>
<td>18.35</td>
<td>67.20</td>
<td>96.50</td>
<td>12.94</td>
<td>18.32</td>
<td>67.23</td>
<td>96.51</td>
</tr>
<tr>
<td>1200</td>
<td>0.00</td>
<td>0.00</td>
<td>2.50</td>
<td>13.05</td>
<td>0.00</td>
<td>0.00</td>
<td>2.53</td>
<td>13.11</td>
<td>0.00</td>
<td>0.00</td>
<td>2.45</td>
<td>13.02</td>
</tr>
</tbody>
</table>

| Monte Carlo Simulation |     |     |     |     |     |     |     |     |     |     |     |     |
| 100                |     |     |     |     |     |     |     |     |     |     |     |     |
| 500                |     |     |     |     |     |     |     |     |     |     |     |     |
| 600                |     |     |     |     |     |     |     |     |     |     |     |     |
| 837.17             |     |     |     |     |     |     |     |     |     |     |     |     |
| 1200               |     |     |     |     |     |     |     |     |     |     |     |     |

| Combined           |     |     |     |     |     |     |     |     |     |     |     |     |
| 100                |     |     |     |     |     |     |     |     |     |     |     |     |
| 500                |     |     |     |     |     |     |     |     |     |     |     |     |
| 600                |     |     |     |     |     |     |     |     |     |     |     |     |
| 837.17             |     |     |     |     |     |     |     |     |     |     |     |     |
| 1200               |     |     |     |     |     |     |     |     |     |     |     |     |
Case 2: Empirical Work and Methodology for European Call Options with Dividends (SPX)

SPX is the option contract that has the S&P 500 index as underlying assets. According to CBOE data, SPX is the most actively traded option contract in the US. SPX Options have European exercise style which means that contract will be exercised only at the maturity.

Due to the fact that SPX has a constant dividend yield of (0.0204), it is important to adjust the underlying stock price for the present value of the dividend amount. In order to make adjustments, the dividend amount has to be discounted by using dividend discount model. The dividend discount model equation is \( PV = \frac{D}{r - g} \) (D is total dividend amount, r is risk free rate, and g is dividend growth).

Table 4: Simulation Prices European Call in-the-money with Dividends (SPX)

<table>
<thead>
<tr>
<th>BSM Analytical</th>
<th>General Monte Carlo</th>
<th>Combined</th>
</tr>
</thead>
<tbody>
<tr>
<td>1724.081</td>
<td>1724.513 (I = 100)</td>
<td>1726.850 (I = 100)</td>
</tr>
<tr>
<td></td>
<td>1724.758 (I = 10,000)</td>
<td>1724.694 (I = 10,000)</td>
</tr>
<tr>
<td></td>
<td>1724.059 (I = 100,000)</td>
<td>1724.109 (I = 100,000)</td>
</tr>
<tr>
<td></td>
<td>1724.122 (I = 200,000)</td>
<td>1724.046 (I = 200,000)</td>
</tr>
<tr>
<td></td>
<td>1724.141 (I = 400,000)</td>
<td>1724.118 (I = 400,000)</td>
</tr>
</tbody>
</table>

When SPX option is in-the-money, the number of iterations performed matters, see Table 4. In fact, when I=100 the biggest difference is the difference between the BSM Model (1724.081) and the Combined Model (1726.850) and its value is 0.16%. With an increase in number of iterations, the difference is smaller. For example, when I=400,000 – the absolute difference between the three methods is 0.004 %. So, it is
possible to conclude that the increase in number of iterations can decrease the absolute difference of market value between the three models from 0.16 % to 0.0004 %.

**Chart 2** is used to show the convergence of The General Monte Carlo Method and The Combined Method estimates again due to Law of Large Numbers.

**Chart 2: In-the-Money SPX Option Contract**

![Chart 2: In-the-Money SPX Option Contract](chart_image)

When the SPX Option Contract is at-the-money and the option will expire in three weeks, the difference in estimated market value of contract between the three models is 0.15%. Furthermore, if SPX option is going to expire in 36 weeks – the absolute difference between the option market price estimated by three models is 0.16%. So, the results from **Table 5** suggest that the fair market value of the at-the-money SPX option can be efficiently estimated using all three models since they have similar performance.
### Table 5: Comparative Option Values SPX

S0 = 2223.43, r=1.56%, σ = 4.67%, Iterations = 400,000  
Time to Expiration (in weeks)

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>3</th>
<th>9</th>
<th>18</th>
<th>36</th>
<th>3</th>
<th>9</th>
<th>18</th>
<th>36</th>
<th>3</th>
<th>9</th>
<th>18</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>BSM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>2124.27</td>
<td>2123.82</td>
<td>2124.21</td>
<td>1231.20</td>
<td>2123.62</td>
<td>2123.92</td>
<td>2124.35</td>
<td>2125.19</td>
<td>2123.60</td>
<td>2123.87</td>
<td>2124.09</td>
<td>2125.11</td>
</tr>
<tr>
<td>500</td>
<td>1724.08</td>
<td>1725.38</td>
<td>1727.32</td>
<td>1731.17</td>
<td>1724.14</td>
<td>1725.48</td>
<td>1727.46</td>
<td>1731.38</td>
<td>1724.11</td>
<td>1725.41</td>
<td>1727.44</td>
<td>1977.12</td>
</tr>
<tr>
<td>2223.43</td>
<td>13.45</td>
<td>25.29</td>
<td>38.62</td>
<td>60.58</td>
<td>13.47</td>
<td>25.33</td>
<td>38.69</td>
<td>60.68</td>
<td>13.47</td>
<td>25.33</td>
<td>38.73</td>
<td>60.59</td>
</tr>
<tr>
<td>2600</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.04</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.04</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.04</td>
</tr>
</tbody>
</table>
Case 3: Empirical Work and Methodology for American Call Options with Dividends

In order to valuate American call options that pay dividends, I will use Apple Option Contract data. The General Monte Carlo Method and Least-Squares Monte Carlo Method are used and compared to mid-point of bid-ask spread. Due to the fact that American Options that pay dividends can be exercised anytime until their maturity, it is not possible to use The Black Scholes Model. The dividend discount model has to be used again in order to figure out the current underlying stock price.

By observing data from Table 6, it is possible to notice that when I =100 the Least-Squares Monte Carlo (48.868) is closer to the Mid-point (49.22) than The General Monte Carlo (46.909). As the number of iteration increase, the difference between the Least-Squares Monte Carlo and General Monte Carlo decreases significantly. This can be illustrated graphically by Chart 3.

Table 6: Simulation Prices American Call in-the-money with Dividends (AAPL)

<table>
<thead>
<tr>
<th>Mid –point</th>
<th>General Monte Carlo</th>
<th>LSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.22 (given)</td>
<td>46.909 (I = 100)</td>
<td>48.868 (I = 100)</td>
</tr>
<tr>
<td></td>
<td>48.560 (I = 10,000)</td>
<td>48.837 (I = 10,000)</td>
</tr>
<tr>
<td></td>
<td>48.883 (I = 100,000)</td>
<td>48.815 (I = 100,000)</td>
</tr>
<tr>
<td></td>
<td>48.955 (I = 200,000)</td>
<td>48.831 (I = 200,000)</td>
</tr>
<tr>
<td></td>
<td>48.957 (I = 400,000)</td>
<td>48.894 (I = 400,000)</td>
</tr>
</tbody>
</table>

It is interesting to point out that it takes more iterations for stock options (GOOGLE and AAPL) to converge. Chart 1 and Chart 3 suggests that the minimum number of iterations should be at least 100,000. On the other hand, Chart 2 shows the optimal number of iterations for SPX Index Option Contract is approximately 10,000. So,
it is possible to conclude that index options need less iterations to converge due to the fact that index represents the average value of 500 US companies and it is less volatile than the option on particular underlying stock such as Google or Apple.

**Chart 3: In-the-Money AAPL Option Contract**

Through this empirical work, it is shown that The General Monte Carlo performs equally efficient as the other two models for a European call option that pays dividends and American call options that do not pay dividends. On the other hand, The General Monte Carlo performs the worst in comparison to the Least-Squares Monte Carlo Method in the valuation of American call options that pay dividends. Furthermore, it is shown that the number of iterations have an impact on the theoretical value of The Monte Carlo Method, The Combined Method and The Least-Squares Monte Carlo. In general, it is possible to notice that all those methods benefit with the increase in the number of iterations.
Conclusion

The main aim of this thesis was to investigate the use of the four different valuation models in pricing European and American call options. I have used Google Option Contract, SPX Contract and Apple Contract in order to conduct the valuation experiments. I observed three different cases and the results suggest that The Black Scholes Model, The General Monte Carlo Method and The Combined Model perform equally well for European Option Contract with dividends (SPX) and American Option Contract with no dividends (GOOGLE). On the other hand, I have found that The Least-Squares Monte Carlo is a better estimator of the theoretical value of American call options with dividends (APPLE) in comparison to The General Monte Carlo Method. Furthermore, it is important to emphasize that the number of iterations plays an important role since it improves significantly option pricing methods. It is shown that valuation algorithms require less iterations for index option (SPX) than for the stock options (10,000 iterations vs. 100,000) due to the lower volatility level of SPX index. The results of pricing models, however, are theoretical values and they might differ from the real market values for particular option contracts. The disadvantage of these pricing model is the failure to incorporate the transaction costs in calculations. Rubinstein (1981) states that “although tests in area of options are not the most efficient, accurate, and conclusive, they have paved the way to better understanding not only of the behavior of option prices but also of stock prices” (p.2). I hope that my research will be a useful to individuals who would like to learn more about option pricing. The greatest limitation to this research was collecting data - option data access is restricted and it requires data purchase. An additional limitation is the fact that I have used The General Monte Carlo Method that assumes that the underlying stock follows the normal distribution. To further my
research, I plan to extend the comparison by using The Geometric Monte Carlo Model since the major assumption of this model is that underlying stock follows a lognormal distribution. Also, I would like to explore the use of The Monte Carlo Method in valuation of Asian and other exotic option contracts.
Notes

1 A computer comes equipped with a random number generator, (usually the command `rand`, which produces a number which is uniformly distributed in [0, 1].
2 Dynamic programming is a method for solving a complex problem by breaking it down into a collection of simples sub-problems.
3 A linear congruential generator is an algorithm that yields a sequence of pseudorandomized numbers.
4 Type of probability distribution in statistics.
5 The root-mean-square error is a frequently used measure of the difference between values (sample and population values).
6 A Callable bond can be defined as a bond that allows issuer to buy back the bond from the bondholders at pre-specified prices on the pre-specified dates (author: Ding).
7 BIM is introduced by G. N. Milstein, E. Platen and H. Schurz in 1998. BMM is introduced by C. Kahl and H. Schurz in 2006. ETD method is considered to simulate the square-root diffusion.
References


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doi:10.1057/palgrave.rm.8250017


doi:10.1142/9789812701022_0008


Appendix

```python
# Monte Carlo Valuation of European Call and
# American call (no div) by using Numpy package(vectorization)
# Code originally created by Yves Hilpisch
# Code can be found in book (Python for Finance) edition 2014

import math
import numpy as np
from time import time
np.random.seed(20000)
t0 = time()

# Parameters used
S0 = 2223.43  # Underlying stock price
K = 2600  # Strike price
T = 1.0  # Time to expiration
r = 0.0155  # Risk-Free interest rate
sigma = 0.0467  # Volatility
M = 50  # Number of Time step intervals
dt = T / M;  # time interval length
I = 400000  # total number of iterations

# Simulating I paths with M time steps
S = np.zeros((M + 1, I))
S[0] = S0
for t in range(1, M + 1):
    z = np.random.standard_normal(I)  # generating pseudo random number
    S[t] = S[t - 1] * np.exp((r - 0.5 * sigma ** 2) * dt + sigma * math.sqrt(dt) * z)

# Calculating the Monte Carlo estimator
C0 = math.exp(-r * T) * np.sum(np.maximum(S[-1] - K, 0)) / I
print(C0)
```
# Black-Scholes-Merton Model Analytical model with Numpy package
# Code originally created by Yves Hilpisch
# Code can be found in book (Python for Finance) edition 2014

```python
import numpy as np

S0 = 2223.43  # Underlying stock price
K = 2600  # Strike price
T = 0.083  # Time to expiration
r = 0.0156  # Risk-free interest rate
sigma = 0.0467  # Volatility

from math import log, sqrt, exp
from scipy import stats

S0 = float(S0)

d1 = (log(S0 / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * sqrt(T))

d2 = (log(S0 / K) + (r - 0.5 * sigma ** 2) * T) / (sigma * sqrt(T))

value = (S0 * stats.norm.cdf(d1, 0.0, 1.0) - K * exp(-r * T) * stats.norm.cdf(d2, 0.0, 1.0))

# stats.norm.cdf -> cumulative distribution function for normal distribution
# print statement for the value
```

# Combination of Monte Carlo Method and Black-Scholes-Merton Model
# Used to evaluate only European call and American call (no div) options
# Code originally created by Yves Hilpisch
# Code can be found in book (Python for Finance) edition 2014

```python
import numpy as np

S0 = 2223.43  # Underlying Stock Price
K = 2600  # Strike Price
T = 0.25  # Time until expiration
r = 0.0156  # Risk-free interest rate
sigma = 0.0467  # Volatility
I = 400000  # Iteration number

# Valuation Algorithm
z = np.random.standard_normal(I)  # generating pseudorandom numbers
ST = S0 * np.exp((r - 0.5 * sigma ** 2) * T + sigma * np.sqrt(T) * z)

# index values at maturity
hT = np.maximum(ST - K, 0)  # inner values at maturity
C0 = np.exp(-r * T) * np.sum(hT) / I  # Monte Carlo estimator

print(C0)
```
```python
# LISN Model
# Code originally created by Yves Hilpisch - Code can be found in book (Python for Finance) edition 2014
import numpy as np
import numpy.random as npr

def gbm_ms_mer(S, option='call'):
    S0 = 100.0
    T = 1.0
    r = 0.05
    sigma = 0.2
    K = 100.
    dt = T/N
    n = 40
    df = np.exp((-r * dt)
    def gen_sn(M, T, anti_paths=True, no_match=True):
        if anti_paths is True:
            sn = npr.standard_normal((M + 1, T / 2))
            sn = np.concatenate((sn, -sn), axis=1)
        else:
            sn = npr.standard_normal((M + 1, T))
        if no_match is True:
            sn = (sn - sn.mean()) / sn.std()
        return(sn)

    # Simulation of index levels
    S0 = S0
    S = gen_sn(M, T)
    S[0] = S0
    Ssn = gen_sn(M, T)
    for t in range(1, M + 1):
        S[t] = S[t - 1] * np.exp((r - 0.5 * sigma ** 2) * dt + sigma * np.sqrt(dt) * sn[t])

    # Case-based calculation of payoff
    if option == 'call':
        h = np.maximum(S - K, 0)
    else:
        h = np.maximum(K - S, 0)

    # LSE algorithm
    V = np.copy(h)
    for t in reversed(range(M, 0, -1)):
        V[t] = np.maximum(V[t + 1] * df, 0)
        C = np.polyval(V[t], S[t])
    S0 = df * S0 / np.sum(V[1])
    return(S0)
```

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