Perfect Polynomials
modulo 2

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Outline

1. Definitions
   - modulo 2
   - What is "Perfect"?
   - Perfect Polynomials
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1 Definitions
   - modulo 2
   - What is "Perfect"?
   - Perfect Polynomials

2 Previous Research
   - Others’
   - Ours

Ugur Caner Cengiz
Perfect Polynomials
Student Symposium 2015
1 Definitions
   • modulo 2
   • What is "Perfect"?
   • Perfect Polynomials

2 Previous Research
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3 Our Research
   • The Program
   • Main Results
   • Speed!!!
What do you mean by "modulo 2"?

In simple terms, ‘a mod b’ gives the remainder when integer a is divided by non-zero integer b.
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What do you mean by "modulo 2"?

- In simple terms, ‘$a \mod b$’ gives the remainder when integer $a$ is divided by non-zero integer $b$.

- Therefore, mod 2 is very simple: if the number is odd, then it is equivalent to 1 and if even, then it is equivalent to 0.

- $13 \equiv 1$ and $54678 \equiv 0 \pmod{2}$
Perfect Numbers

Sigma function $\sigma$

**Definition**

Lower case Greek letter sigma ($\sigma$) symbolizes an arithmetic function that sums the positive divisors of a positive integer.

$$\sigma(n) = \sum_{d|n} d$$

**Definition**

If $\sigma(n) = 2n$, then $n$ is perfect.

**Example**
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Example
- 6
- $\sigma(6) = 1 + 2 + 3 + 6 = 12$
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- 6
- $\sigma(6) = 1 + 2 + 3 + 6 = 12$
- Hence, 6 is perfect.
Continuing on $\sigma$

Theorem

$\sigma$ is multiplicative over integers.
If $\gcd(m,n) = 1$, then $\sigma(mn) = \sigma(m) \times \sigma(n)$

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- $\sigma(728) = 1 + 2 + 4 + 7 + 8 + 13 + 14 + 26 + 28 + 52 + 56 + 91 + 104 + 182 + 364 + 728 = 1680$
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- So, $\sigma(2^3) \times \sigma(7) \times \sigma(13) = (8 + 4 + 2 + 1)(7 + 1)(13 + 1)$
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- So, $\sigma(2^3) \times \sigma(7) \times \sigma(13) = (8 + 4 + 2 + 1)(7 + 1)(13 + 1)$
- $\sigma(2^3) \times \sigma(7) \times \sigma(13) = 15 \times 8 \times 14 = 1680$
Polynomials mod 2

- For a polynomial mod 2, the coefficients are mod 2. Thus,
  \[5x^7 + 9x^6 + 16x^5 + 48x^4 + x^3 + 4x^2 + 71x + 1 \equiv x^7 + x^6 + x^3 + x + 1\]

\(x^2 + 2x + 1^2 \equiv x^2 + 1 \pmod{2}\)
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- For example, $x^2 + x = x \times (x + 1)$
- $\sigma(x^2 + x) = (x^2 + x) + (x + 1) + x + 1 = x^2 + 3x + 2$
- $x^2 + 3x + 2 \equiv x^2 + x \pmod{2}$
- So $\sigma(x^2 + x) = x^2 + x \pmod{2}$
- $x^2 + x$ is a perfect polynomial mod 2.
E. F. Canaday

Gallardo and Rahavandrainy
Perfect polynomials mod 2 exist in two ways:
\[ x^h(x + 1)^k A \text{ and } B^2, \] where B is relatively prime to \( x(x + 1) \)

Gallardo and Rahavandrainy

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<table>
<thead>
<tr>
<th>Degree</th>
<th>Factorization into Irreducibles</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$T(T + 1)^2(T^2 + T + 1)$</td>
</tr>
<tr>
<td></td>
<td>$T^2(T + 1)(T^2 + T + 1)$</td>
</tr>
<tr>
<td>11</td>
<td>$T(T + 1)^2(T^2 + T + 1)^2(T^4 + T + 1)$</td>
</tr>
<tr>
<td></td>
<td>$T^2(T + 1)(T^2 + T + 1)^2(T^4 + T + 1)$</td>
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</table>

**Figure**: Canaday’s list for perfects
Check if $\sigma B = B$. Output $B$. 
The Algorithm to Find the Perfect Polynomials

- Check if $\sigma B = B$. Output $B$.
- If not, compute $D$ where $D = \sigma(B) / \gcd(B, \sigma(B))$
The Algorithm to Find the Perfect Polynomials

- Check if $\sigma B = B$. Output $B$.
- If not, compute $D$ where $D = \frac{\sigma(B)}{\gcd(B, \sigma(B))}$
- If $\gcd(B, D) > 1$, then stop. No output!

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The Algorithm to Find the Perfect Polynomials

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- If $\gcd(B, D) > 1$, then stop. No output!
- If the polynomial passes the test on step 3, then let $P$ be the greatest factor of $D$. 
Check if $\sigma B = B$. Output $B$.

If not, compute $D$ where $D = \sigma(B) / \gcd(B, \sigma(B))$.

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If the polynomial passes the test on step 3, then let $P$ be the greatest factor of $D$.

Restart the algorithm taking $BP, BP^2, BP^3, ..., BP^k$ where degree of $BP^k < K$. 
def primPerf(B):
    if B == sumDivs4(B):
        return B
    else:
        D = (sumDivs4(B)/gcd(B, sumDivs4(B)))
        if gcd(D,B) != 1:
            return False
        else:
            F = D.factor()
            P = F[len(F)-1][0]
            check = False
            K = 1
            while (B*(P^K)).degree() <= 1000:
                check = primPerf(B*(P^K))
                if check == False:
                    K = K + 1
                else:
                    return primPerf((B*(P^K)))
            break
Results up to degree 200

\[ x \times (x + 1)^2 \times (x^2 + x + 1) \]
\[ x \times (x + 1)^2 \times (x^2 + x + 1)^2 \times (x^4 + x + 1) \]
\[ (x + 1) \times x^2 \times (x^2 + x + 1) \]
\[ (x + 1) \times x^2 \times (x^2 + x + 1)^2 \times (x^4 + x + 1) \]
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\[ x^4 \times (x + 1)^4 \times (x^4 + x^3 + 1) \times (x^4 + x^3 + x^2 + x + 1) \]
\[ x^4 \times (x + 1)^6 \times (x^3 + x + 1) \times (x^3 + x^2 + 1) \times (x^4 + x^3 + x^2 + x + 1) \]
\[ (x + 1)^3 \times x^6 \times (x^3 + x + 1) \times (x^3 + x^2 + 1) \]
\[ (x + 1)^4 \times x^6 \times (x^3 + x + 1) \times (x^3 + x^2 + 1) \times (x^4 + x^3 + 1) \]
\[ x \times (x + 1) \]
\[ x^3 \times (x + 1)^3 \]
\[ x^7 \times (x + 1)^7 \]
\[ x^{15} \times (x + 1)^{15} \]
\[ x^{31} \times (x + 1)^{31} \text{ and } x^{63} \times (x + 1)^{63} \]
def sigma1(x, y):
    return \( \frac{x^{y+1} - 1}{x - 1} \)

def sigma2(x, y):
    sum = 0
    for pow in range(0, y + 1):
        sum = sum + \( x^{pow} \)
    return sum

**sigma1 and sigma2 speed testing**

Dynamic Programming
Our Research

Figure: FAST!

```python
import time
tic = time.clock()
sum = x^30
found = primPerf(sum)
if type(found) == type(x):
    print found, "="
    found.factor()
toc = time.clock()
toc - tic

perfFinder(15)

1 = 1
degree = 0

x^2 + x = x * (x + 1)
degree = 2

x^2 + x = x * (x + 1)
degree = 2
```
A perfect polynomial equals the sum of its divisors.

As Canaday thought there are no odd perfect polynomials up to degree 200.

My program is relatively fast and finds the perfect polynomials.

Future Plans

- To check higher degrees
- Show odd perfect polynomials mod 2 have at least 6 factors
- Work on a paper
For Further Information

E.F. Canaday
*The Sum of The Divisors of a Polynomial.*
*Duke Mathematical Journal, 8(4):721–737, 1941*

L. Gallardo. and O. Rahavandrainy.
*Odd Perfect Polynomials over $F_2$*

L. Gallardo. and O. Rahavandrainy.
*There is no odd perfect polynomial over $F_2$ with four prime factors*
Thank You!

(Any Questions?)