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LAKE FOREST COLLEGE

Senior Thesis

Applications of Algebraic Topology

by

Jason Veenendaal

April 25, 2014

The report of the investigation undertaken as a
Senior Thesis, to carry one course of credit in
the Department of Mathematics

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Michael T. Orr
Krebs Provost and Dean of the Faculty

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Abstract

This thesis will thoroughly examine the basic ideas of homotopy theory and follow this with a treatment of the fundamental group and several other important results. The goal of this thesis is to use these ideas and show my understanding of them to prove the fundamental theorem of algebra and the Brouwer fixed-point theorem for the disc.

1 Introduction

Topology is the study of topological spaces and continuous functions between them [2]. Classifying spaces based on homeomorphisms, which is the intuitive notion of one space being continuously transformed into another, is of prime importance for algebraic topology. Algebraic topology attempts to study this by using ideas and techniques from abstract algebra such as groups and homomorphism.

This thesis will examine the ideas of the fundamental group which will be used to prove two results-the fundamental theorem of algebra and the Brouwer fixed-point theorem for the disc. Most of the work done on homotopy theory was done by the mathematician L. E. J. Brouwer whose theorem we will later prove [3].

Unless otherwise stated, all theorems in this thesis were extracted from Munkres' textbook on topology, see [1].

2 Definitions

In this thesis, we will deal with topological spaces and continuous maps or continuous functions. We define a topological space to be a set with a topology. A topology on a set X is a collection of subsets T of X called open sets that satisfy the following properties. The empty set and X are open sets. The union of the elements of any subcollection of T is an open set. The intersection of the elements of any finite subcollection of T is an open set.

We say that a map or function $f : X \to Y$ is continuous if for any open set U in Y, $f^{-1}(U)$ is an open set in X. We say that a function f is a homeomorphism if it is a bijection (one-to-one and onto) and is continuous in both directions. This means that f is continuous and f^{-1} is continuous.

Let f and f' be continuous functions from $X \to Y$, where X and Y are topological spaces. We say f and f' are homotopic to each other if $\exists F : X \times [0, 1] \to Y$, a continuous function, $\ni F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$. F is called a homotopy between f and f'. We call $I = [0, 1]$ for shorthand. If f is homotopic to f', we write $f \simeq f'$.

We define a path. A path is a continuous function f where $f : I \to X$ where X is a topological space.

If f and f' are paths $\ni f : I \to X$, $f' : I \to X$ where $f(0) = f'(0) = x$ and $f(1) = f'(1) = y$, then f and f' are path homotopic if $\exists F : I \times I \to X$, a continuous function, $\ni F(s, 0) = f(s), F(s, 1) = f'(s)$, $F(0, t) = x$, $F(1, t) = y$. We call F a path homotopy between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

If f and f' map X into $\mathbb{R}^n$, we can create a homotopy between them where $F(x, t) = (1 - t) * f(x) + t * f'(x)$. This is called the straight line homotopy. In general, this straight line homotopy always exists for $f, f' : X \to Y$ when Y is a convex set, which is a subset of a vector space such that $\forall a, b \in Y$ and $\forall t \in [0,1], ta + (1-b)t \in Y$. Moreover, two maps into a convex set are always homotopic.

3 Examples

Let $f : \mathbb{R} \to \mathbb{R}$ and $f' : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$ and $f'(x) = 2x$. Define $F(x, y) = (1 + y) * x$. So, $F(x, 0) = x = f(x)$ and $F(x, 1) = 2x = f'(x)$. $F(x, y)$ is continuous on $\mathbb{R} \times I$. So, F is a homotopy between f and f' which means f and f' are homotopic.

Let $f : I \to I$ and $f' : I \to I$. Define $f(x) = x$ and $f'(x) = x^2$. $f(0) = f'(0) = 0$ and $f(1) = f'(1) = 1$. Let F use the straight line homotopy so $F(x, y) = (1 - y) * x + y * x^2$. We verify that $F(0, y) = 0$, $F(1, y) = 1$, $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$. Thus, since F is continuous, we have shown that f and f' are path homotopic because there exists such a function, F.

4 Algebraic Properties

Theorem 4.1. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. To prove that a relation is an equivalence relation, we must show that the relation is transitive, reflexive, and symmetric. Transitive means that if $f \simeq f'$ and $f' \simeq f''$ then $f \simeq f''$. Reflexive means that $f \simeq f$ or that f is related to itself. Symmetric means that if $f \simeq f'$ then $f' \simeq f$. Thus, we must prove that $\simeq$ and $\simeq_p$ hold these three properties to show that they are equivalence relations.

We prove that $\simeq$ and $\simeq_p$ are transitive. Let $f \simeq f'$ and $f' \simeq f''$. Let F be a homotopy between f and f' and let F' be a homotopy between f' and f''. Define a function $G : X \times I \to Y$ by the equation

$$G(x, t) = \begin{cases} F(x, 2t) & : t \in [0, \frac{1}{2}] \\ F'(x, 2t - 1) & : t \in [\frac{1}{2}, 1] \end{cases}$$

This new map is well defined and continuous since if $t = \frac{1}{2}$, we have $F(x, 2t) = f'(x) = F'(x, 2t - 1)$. However, we have just defined a homotopy between f and f'' since $G(x, 0) = F(x, 0) = f(x)$ and $G(x, 1) = F'(x, 1) = f''$. Thus, $f \simeq f''$. Moreover, this homotopy holds if F and F' are path homotopies. This shows that G is a path homotopy since we define $G : I \times I \to Y$ and use the same equation.

We prove that $\simeq$ and $\simeq_p$ are reflexive. If f is a continuous map, it is trivial to show that it is homotopic to itself because we can define the homotopy $F(x, t) = f(x)$ for all t which shows that $f \simeq f$. This same homotopy works if f is a path because a path is a continuous function.

We prove that $\simeq$ and $\simeq_p$ are symmetric.Let $f \simeq f'$, then there exists a homotopy F between them. We define the homotopy $G(x,t) = F(x, 1-t)$. It follows that for $t = 0$, $G(x, 0) = F(x, 1) = f'(x)$ and for $t = 1$, $G(x, 1) = F(x, 0) = f(x)$ which shows that $f' \simeq f$. The same logic works if f and f' are paths.

This completes the proof that $\simeq$ and $\simeq_p$ are both equivalence relations. We call $[f]$ the path homotopic equivalence class of f which is the set of all elements that are path homotopic to f.

□

5 Groupoid

First, we recall the definition of a group and a binary structure. We define a binary structure to be on a set with an operation $*$ where for any two elements in the set a and b, we define $a * b = c$ where c is some element of the set. Let $< S, * >$ be a binary structure on set S. It is a group if the operation $*$ is associative such that $\forall a, b, c \in S$, $a * (b * c) = (a * b) * c$, there exists an identity element $e \in S$ such that $\forall a \in S$, $a * e = e * a = a$, and there exists inverse elements a' such that $\forall a \in S$, $a * a' = a' * a = e$.

We will define a groupoid as holding all the same properties of a group except that the operation $*$ is not defined for all pairs of elements in the set. This section will prove that the operation $*$ on equivalence classes of paths is a groupoid. We define $*$ as follows.

If f is a path that starts at x and ends at y and g is a path that starts at y and ends at z, we define the operation $f * g = h$ where h is given by the equation

$$h(s) = \begin{cases} f(2s) & : s \in [0, \frac{1}{2}] \\ g(2s-1) & : s \in [\frac{1}{2}, 1] \end{cases}$$

Theorem 5.1. *The operation $[f] * [g] = [f * g]$ is well defined.*

Proof. Let f and f' be any elements of $[f]$ and g and g' be any elements of $[g]$. We want to show $f * g \simeq_p f' * g'$ which is sufficient to show that this operation is well defined. Let F be the path homotopy between f and f' and let G be the path homotopy between g and g'. Define

$$H(s,t) = \begin{cases} F(2s,t) & : s \in [0, \frac{1}{2}] \\ G(2s-1,t) & : s \in [\frac{1}{2}, 1] \end{cases}$$

It is easy to see that $H(s,t)$ is a path homotopy between $f * g$ and $f' * g'$. We can check by setting $t = 0$ to see that $H(s, 0)$ will be equal to f on the first half of the interval and equal to g on the second half. Likewise, we can see that $H(s, 1)$ is equal to f' on the first half of the interval and equal to g' on the second half of the interval. This is exactly what we needed for a homotopy between $f * g$ and $f' * g'$. □

With that, we will now focus on the equivalence classes as elements instead of the individual paths due to some more interesting properties defined for the equivalence classes.

The operation $*$ on equivalence classes of paths in X, some topological space, has three important properties that we define as being a groupoid. We cannot call it a group because the operation $*$ is not defined for paths f and g where $f(1) \neq g(0)$ despite meeting other requirements for being a group. We prove the following three properties.

Theorem 5.2. *The operation $*$ is associative in that $[f] * ([g] * [h]) = ([f] * [g]) * [h]$ whenever the operations are defined.*

Let e_x be the constant path where $e_x : I \to X$ where $e_x(s) = x$ for all $s \in I$ and let f be a path from x_0 to x_1. Then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

Let f be a path in X from x_0 to x_1 and say $\bar{f}$ is defined as $\bar{f}(s) = f(1 - s)$. We say $\bar{f}$ is the reverse of f and $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

Proof. First, we must note two rather elementary properties about path homotopies. The first is that given a continuous map $k : X \to Y$ and a path homotopy F in X between f and f', $k \circ F$ is a path homotopy in Y between $k \circ f$ and $k \circ f'$. By definition of a homotopy, $F(s, 0) = f(s)$ which implies that $k \circ F(s, 0) = k \circ f(s)$. Also, $F(s, 1) = f'(s)$ which implies that $k \circ F(s, 1) = k \circ f'(s)$. This is exactly the definition of a homotopy between two paths since $k \circ F$ maps $I \times I \to Y$ and both $k \circ f$ and $k \circ f'$ are paths into Y.

The second elementary property is that given a continuous map $k : X \to Y$ and paths f and g in X with $f(1) = g(0)$ then

$$k \circ (f * g) = (k \circ f) * (k \circ g)$$

This is easy enough to show since, by definition, $k \circ (f * g)$ is equivalent to $k \circ f$ on the first half of the interval and equivalent to $k \circ g$ on the second half of the interval. This is the same definition of $(k \circ f) * (k \circ g)$.

We will first verify the second property. We will let e_0 be the constant path in I where all points get mapped to the point 0. Then consider the identity map of I given as i so that $e_0 * i$ is also a path in I from 0 to 1. Since I is a convex space, there exists the straight line path homotopy between i and $e_0 * i$ which we will call G. By previously stated identity, $f \circ G$ is a path homotopy between $f \circ i = f$ and

$$f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f$$

which is just the constant map of the starting point of f operated with f. Thus, f is homotopic with $e_{x_0} * f$. Thus, $[f] = [e_{x_0} * f] = [e_{x_0}] * [f]$ which is what we wanted to show. We can do the same style agrument to prove that $[f] * [e_{x_1}] = [f]$. We simply show that $i * e_1$ is homotopic to i in I since I is convex. Then we use the same identity that $f \circ (i * e_1) = (f \circ i) * (f \circ e_1) = f * e_{x_1}$ which is homotopic to $f \circ i = f$. Thus, $[f * e_{x_1}] = [f] * [e_{x_1}] = [f]$. We have then proved the second property.

For the third property, we will consider the reverse map of the identity map i which is $\bar{i}(s) : 1 - s$. Then $i * \bar{i}$ is a path in I that begins and ends at 0 as well as the constant map e_0 so they are path homotopic. We will call their homotopy H so that $f \circ H$ is a homotopy between $f \circ e_0 = e_{x_0}$ and

$$(f \circ i) * (f \circ \bar{i}) = f * \bar{f}$$

Thus, $[e_{x_0}] = [f * \bar{f}] = [f] * [\bar{f}]$. Similarily, we can show that $[e_{x_1}] = [\bar{f}] * [f]$. Thus, we have proved the third property.

For the final part of the proof, we will describe the $f * g$ in a different way. If $[a, b]$ and $[c, d]$ are two intervals in $\mathbb{R}$, then there is a unique map $p : [a, b] \to [c, d]$ of the form $p(x) = mx + k$ which maps a to c and b to d. We call this the positive linear map of $[a, b]$ to $[c, d]$ which is a straight line with positive slope. The inverse of a positive linear map is also a positive linear map as well as the composite of two positive linear maps.

We can now describe $f * g$ as the positive linear map of $[0, \frac{1}{2}]$ to $[0, 1]$ followed by f and the positive linear map of $[0, \frac{1}{2}]$ to $[0, 1]$ followed by g. We will consider the triple product of paths $f * (g * h)$ and $(f * g) * h$ and assume that $f(1) = g(0)$ and $g(1) = h(0)$ so that operation is defined for these paths in X. We will choose points a and b of I such that $0 < a < b < 1$. Fix paths f,g,h and define a path $k_{a,b}$, which depends on f, g, h, in X as follows: on $[0, a]$ it equals the positive linear map of $[0, a]$ to I followed by f; on $[a, b]$ it equals the positive linear map of $[a, b]$ to I followed by g; and on $[b, 1]$ it equals the positive linear map of $[b, 1]$ to I followed by h. It is important to note that $k_{a,b}$ will change depending on the choice of a and b. We will show then that another path $k_{c,d}$ is path homotopic to $k_{a,b}$.

To show this, we will consider a map $p : I \to I$ that is equal to the composition of the positive linear maps $[0, a]$ to $[0, c]$, $[a, b]$ to $[c, d]$, and $[b, 1]$ to $[d, 1]$. It follows then that $k_{c,d} \circ p = k_{a,b}$. We know that p is a map in I from 0 to 1 as well as the identity map i. Since I is convex, there exists a path homotopy in I called P between the two paths. Then, $k_{c,d} \circ P$ is a path homotopy in X between $k_{c,d} \circ p = k_{a,b}$ and $k_{c,d} \circ i = k_{c,d}$. This shows us that $[k_{c,d}] = [k_{a,b}]$ for any a, b, c, d between 0 and 1. So if $a = \frac{1}{2}, b = \frac{3}{4}, c = \frac{1}{4}$ and $d = \frac{1}{2}$, we have that $[f * (g * h)] = [K_{\frac{1}{2},\frac{3}{4}}] = [K_{\frac{1}{4},\frac{1}{2}}] = [(f * g) * h]$. This simplifies to $[f] * ([g * h]) = [f] * ([g] * [h]) = ([f * g]) * [h] = ([f] * [g]) * [h]$ which is exactly what we wanted to prove. Thus, the last part is proved so all there properties are shown to be true. $\square$

We use the operation $*$ on $f : I \to X$ and $g : I \to X$ if $f(1) = g(0)$. This operation used on equivalence classes forms a groupoid because it is not defined for all continuous functions from $I \to X$ as well as lacking a singular identity element for all paths. This is because the left identity is created by taking the equivalence class of a constant path that begins at the start point of that path. The right identity is created by taking the equivalence class of a constant path that begins at the end point of that path. Since paths can start and end at different points, we can have different identity elements. Otherwise, it holds the same properties of a group.

6 The Fundamental Group

As stated, the operation $*$ is not defined for all paths in X since two paths f and g might not be defined for $f * g$ if $f(1) \neq g(0)$. Thus, not every path in X can be operated with each other. This is why we call the operation $*$ on equivalence classes of paths in X a groupoid and not a group.

However, we can overcome this problem by fixing a point, x_0, in X and considering the paths that start and end at this point. We call these paths loops in X. If we look at only the loops based at this point, we call the equivalence classes of all such loops the fundamental group of X fixed at the point x_0.

It can now be shown that this new set of equivalence classes with operation $*$ form a group by the previous conditions listed. Every loop in this set can be operated with each other since they all start and end at the same point, x_0. We now have a group.

Theorem 6.1. *We will denote this group by $\pi_1(X, x_0)$ where x_0 is the base point. We will consider a path α in X from x_0 to x_1. We define the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the equation $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$ where $\bar{\alpha}$ is the inverse map of α. The map $\hat{\alpha}$ is a group isomorphism.*

Proof. We first must show that $\hat{\alpha}$ is a homomorphism. We simply compute that $\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha])$. Which simplifies to $[\bar{\alpha}] * [f] * [g] * [\alpha] = \hat{\alpha}([f] * [g])$ which is exactly what we needed to prove.

To show that $\hat{\alpha}$ is an isomorphism, we show that if β denotes the path $\bar{\alpha}$, which is the reverse of α, then $\hat{\beta}$ is an inverse for $\hat{\alpha}$. Consider an element h of $\pi_1(X, x_1)$, then $\hat{\beta}([h]) = [\hat{\beta}]*[h]*[\beta] = [\alpha]*[h]*[\hat{\alpha}]$. Thus, $\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] = [h]$. We can similarily show that $\hat{\beta}(\hat{\alpha}([f])) = [f]$ for each $[f] \in \pi_1(X, x_0)$.

□

If the space X is path connected, that is, there exists such a path α between any two points in X, then any two $\pi_1(X, x)$ in X are isomorphic.

A space X is simply connected if it is path connected and $\pi_1(X, x_0)$ for some x_0 in X is the trivial group (meaning it only has one element in it which must be the identity). This implies that then every $\pi_1(X, x)$ for any $x \in X$ is the trivial group since X is path connected and thus $\pi_1(X, x)$ is isomorphic to $\pi_1(X, x_0)$. That is, every loop is homotopic to a constant loop.

We will also consider a homomorphism induced by h where h is a continous map from a space X to a space Y. We first have to fix a base point in X which we call x_0. Thus, $h(x_0) = y_0 \in Y$. We denote such an h by, $h : (X, x_0) \to (Y, y_0)$, we define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $h_*([f]) = [h \circ f]$. We must be careful to distinguish which point is being mapped to which point. We can write $(h_{x_0})_*$ as the map for base point x_0 and $(h_{x_1})_*$ as the map for x_1 because even if X is path connected, the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are still different groups.

We can show that this map h_* is a homomorphism based on the previously discussed identity

$$(h \circ f) * (h \circ g) = h \circ (f * g)$$

We can then cleary see how this map is a homomorphism since the operation $*$ is defined on both fundamental groups. This map is also well defined since if F is a path homotopy between the paths f and f', then $h \circ F$ is path homotopy between $h \circ f$ and $h \circ f'$ by our previous identity.

We must also define a homeomorphism. Let f be a continuous map from X to Y. If f is bijective and its inverse map, f^{-1}, is continuous, we call f a homeomorphism between X and Y.

Theorem 6.2. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof. By definition,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f]$$

$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)]$$

and since the composition of maps is associative we have that $(k \circ h)_* = k_* \circ h_*$. Likewise, $i_*([f]) = [i \circ f] = [f]$. Thus, the proof is complete. □

The power of this proof comes more so from its corollary.

Theorem 6.3. *If $h : (X, x_0) \to (Y, y_0)$ is homeomorphism of X and Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.*

Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h which we know exists and is continous since h is a homeomorphism. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0) and itself and $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the identity map of (Y, y_0). This means that i_* is the identity homomorphism of the group $\pi_1(X, x_0)$ and j_* is the identity homomorphism of the group $\pi_1(Y, y_0)$. Because of this, we know that h_* and k_* must be inverses of each other since composing on either side gives of an identity map. This means that h_* has a inverse in k_* which means that h_* is an isomorphism. □

The power of this corollary is in its contrapositive which states that if two spaces have different or rather non-isomorphic fundamental groups, they cannot possibly be homeomorphic with each other. This means we can simply determine if two spaces can even be homeomorphic by computing their fundamental groups. This task can be somewhat hard to do but there are several ways to compute them. One of the ways to compute them is to use covering spaces.

7 Covering Spaces

We define S^1, the unit circle, as $\{(x, y) : x^2 + y^2 = 1\}$.

Let $p : E \to B$, where E and B are topological spaces, be a continuous surjective (onto) map. The open set U of B is said to be evenly covered by p if the inverse image of $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. We call the collection of V_α as the parition of $p^{-1}(U)$ into slices.

Following this, we say p is a covering map if every point b in B is contained in an open neighborhood U that is evenly covered by p. If this is true, we call E a covering space of B.

Later on, it will become useful to look at the covering spaces of the unit circle S^1. We will first examine a map $p : \mathbb{R} \to S^1$ given by the equation $p(x) = (\cos 2\pi x, \sin 2\pi x)$. We will need to prove that this is indeed a covering map of S^1.

Theorem 7.1. *The map $p : \mathbb{R} \to S^1$ given by the equation $p(x) = (cos 2\pi x,\ sin 2\pi x)$ is a covering map of S^1.*

Proof. To prove this we must show that each point in S^1 is contained in an open set that is evenly covered by $\mathbb{R}$. We will divide the unit circle into four overlapping parts. The first part will be the open set containing only positive x coordinates. The second part will be the open set containing only negative x coordinates. The third and fourth parts will be the open sets containing positive and negative y coordinates respectively. We can see that the union of these four parts is the entire unit circle so we must only prove that each part is evenly covered by p.

To do this, we first must prove that if $f : A \to B$, f is continuous surjective and A is compact, then $f(A)$ is compact. Also, we must prove that if A is compact and $f : A \to B$ is continuous and bijective, then f is a homeomorphism.

Let U be an open cover of B. Since f is continuous, then for every open set in U, U_a, $f^{-1}(U_a)$ is open in A. Furthermore, $f^{-1}(U_a) \bigcup f^{-1}(U_b) \bigcup \cdots$ for all open sets in U is an open cover of A since for every $x \in A$, $f(x) \in U_i$ for some $U_i \in U$. Since A is compact, then there exists an finite subcover of $f^{-1}(U_a) \bigcup f^{-1}(U_b) \bigcup \cdots$ name it $f^{-1}(U_a) \bigcup f^{-1}(U_b) \bigcup \cdots \bigcup f^{-1}(U_n)$. It follows then that $U_a \bigcup \cdots \bigcup U_n$ is a finite subcover of B. Thus, for any open cover of B, we have a finite subcover which means B is compact.

Likewise, let f be bijective and continuous. To show that f is a homeomorophism, we must show that f maps open sets to open sets. Since f is bijective, $f(U)^c = f(U^c)$ for some open set U. This means that $f(U^c)$ is compact and, moreover, that $f(U^c)$ is closed. The complement of a closed set is open so $f(U^c)^c$ is open. Since $f(U^c) = f(U)^c$, $f(U)$ is open. Thus, we have shown that f maps open sets to open sets and thus f is a homeomorphism.

Now that we have proven this, we will consider the inverse image of the open set containing only positive x coordinates. The inverse image of such a set takes the form

$$V_n = (n - \frac{1}{4}, n + \frac{1}{4})$$

for every n an element of the integers. Now we will look at these V_n under the closed interval so they contain the points $n - \frac{1}{4}$ and $n + \frac{1}{4}$. This map is injective since the sin $2\pi x$ function is always increasing on this interval. Likewise, this function is surjective since for any point in the set containing the positive x coordinates and 0 x coordinates, there exists a point in the closure of V_n that maps to that point by the intermediate value theorem.

p is bijective and it maps a compact set with the closure of V_n. Then we know from the previous theorem that p must be a homeomorphism when restricted to the closure of V_n since this is when it is bijective. Then p limited to V_n itself is a homeomorphism to the open set containing only the positive x coordinates. Thus, we have shown that p evenly covers this open set with V_n. We can create entirely similar arguments to show the other three open sets are evenly covered by p. Since the union of these four sets contains all the points in the unit circle, we now know that p is a covering map of S^1. □

The fact that p is a covering map will play an important part in computing the fundamental group of the circle which will be the focus of the next chapter.

8 The Fundamental Group of the Circle

Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a lifting of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

An important result about lifting maps is that paths will have unique liftings. This will be immensely important when computing the fundamental group of the circle. We prove this fact now.

Theorem 8.1. *Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.*

Proof. We must first prove existence of the lifting path $\tilde{f}$. Fix a path $f : I \to B$. For each $t \in I, f(t) \in B$, so $\exists$ open $U_t \subset B$ with $f(t) \in U_t$ such that $p^{-1}(U_t)$ are disjoint slices. Since $f(I) \subset \bigcup U_t$ and $f(I)$ is compact, then there exists a finite number of U_t where $f(I)$ is contained in them. Since there are a finite number of them, we divide the intervals of I so that $f([s_i, s_{i+1}])$ is contained in one of these open sets for all $s_i \in [0,1]$. We define the lifting map $\tilde{f}$ using mathematical induction.

First, define $\tilde{f}(0) = e_0$. We can easily see that this works since $p \circ \tilde{f}(0) = f$. Now assume $\tilde{f}$ is defined for $[0, s_i]$. Now define $\tilde{f}$ on $[s_i, s_{i+1}]$. By definition$f[s_i, s_{i+1}]$ is contained in an open set U. This open set U is evenly covered by p so that $p^{-1}(U) = V_\alpha$ where V_α is composed of the union of the disjoint open sets. So, $p^{-1}([s_i, s_{i+1}])$ is contained in one of these open set call it V_0. Since $\tilde{f}(s_i)$ is defined we know that $\tilde{f}(s_i)$ is in this open set V_0. We then define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by the equation

$$\tilde{f}(s) = (p|V_0)^{-1}(f(s))$$

This is helpful since we know that $(p|V_0)^{-1}$ exists and is continuous since $p|V_0$ is a homeomorphism. We can see that this defines $\tilde{f}$ for $[s_i, s_{i+1}]$ and is continuous for $[0, s_{i+1}]$. Since we showed this for an arbitrary interval $[0, S_i]$ in $[0,1]$, by induction, we have proved that $\tilde{f}$ is defined and thus exists on $[0,1]$.

To prove uniqueness, we will use induction again. Suppose $\bar{f}$ is another lifting of f beginning at e_0. Then $\bar{f}(0) = e_0 = \tilde{f}(0)$ so the base case is true. Suppose then that $\bar{f}(s) = \tilde{f}(s)$ for al $s \in [0, s_i]$. Let V_0 be the same open set in E from the previous paragraph, then for $s \in [s_i, s_{i+1}]$, $\tilde{f}(s)$ is defined as $(p|V_0)^{-1}(f(s))$. But what can $\bar{f}(s)$ be? Since $\bar{f}$ is a lifting of f, it must carry the interval $[s_i, s_{i+1}]$ into the set $p^{-1}(U) = \bigcup V_\alpha$. The open sets of V_α are disjoint so that $\bar{f}([s_i, s_{i+1}])$ must lie entirely in one of these open sets. Thus, it must lie entirely in the open set V_0. Thus, for $s \in [s_i, s_{i+1}]$, $\bar{f}(s)$ must equal some point $y \in V_0$ lying in $p^{-1}(f(s))$. There can be only one such point by the definition of being evenly covered. Thus, $\bar{f}(s) = \tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$. Then, by induction, $\bar{f}(s) = \tilde{f}(s)$ for all $s \in [0,1]$. Thus, $\tilde{f}$ is unique. □

Theorem 8.2. *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be* *continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map*

$$\tilde{F} : I \times I \to E$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof. First, we define $\tilde{F}(0,0) = e_0$. Next, we use the preceding theorem to extend $\tilde{F}$ to the left-hand edge $0 \times I$ and the bottom edge $I \times 0$ of $I \times I$. Then we must find a subdivision of $I \times I$ such that each rectangle of this subdivision is mapped by F into an open set of B that is evenly covered by p.

We can do this if we consider for every $x \in I \times I$ a rectangle R_x that contains that point that is a subset of $F^{-1}(G)$ where G is an open set evenly covered by p. Then the union of the open interior of each of the R_x contains $I \times I$. Since $I \times I$ is compact, there exists a finite subcover of rectangles that contains $I \times I$. However, these rectangles may not be disjoint from each other. So, we will slice the rectangles by taking the x endpoints of the rectangles and the y endpoints of the rectangles. If we number these in order, we slice going from the bottom to top at each y endpoint and we slice going left to right at each x endpoint. What we have remaining is a colletion of new rectangles that do not overlap but yet each new rectangle still gets mapped by F into an opet set G that is evenly covered by p since each new rectangle is just a subset of the original rectangles.

Say that this subdivision is numbered

$$s_0 < s_1 < \cdots < s_m$$

$$t_0 < t_1 < \cdots < t_n$$

where the rectangle

$$I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$$

is one of the rectangles previously described. We will proceed with a proof by induction and say that given some i_0 and j_0, we assume that $\tilde{F}$ is defined on the set A which is the union of $0 \times I$ and $I \times 0$ and all the previous rectangles to $I_{i_0} \times J_{j_0}$ (those rectangles $I_i \times J_j$ for which $j < j_0$ and those for which $j = j_0$ and $i < i_0$). Assume also that $\tilde{F}$ is a continuous lifting of $F|A$. We define $\tilde{F}$ on $I_{i_0} \times J_{j_0}$. Choose an open set U of B that is evenly covered by p and contains $F(I_{i_0} \times J_{j_0})$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices; each set V_α is mapped homeomorphically onto U by p. Now $\tilde{F}$ is already defined on the set $C = A \bigcap (I_{i_0} \times J_{j_0})$. This set is the union of the left and bottom edges of the rectangle $I_{i_0} \times J_{j_0}$ since we are defining each previous one going from left to right one row at a time and then moving up a row. C is connected then. Therefore, $\tilde{F}(C)$ is connected. Call V_0 the

V_α the slice of $p^{-1}(U)$ that contains $\tilde{F}(C)$.

Let $p_0 : V_0 \to U$ denote the restriction of p to V_0. Since $\tilde{F}$ is a lifting of $F|A$, we say that for any $x \in C$,

$$p_0(\tilde{F}(x)) = p(\tilde{F}(x)) = F(x),$$

so that $\tilde{F}(x) = p_0^{-1}(F(x))$. Hence, we may extend $\tilde{F}$ by defining

$$\tilde{F}(x) = p_0^{-1}(F(x))$$

for $x \in I_{i_0} \times J_{j_0}$. It is straightforward to see the extended map is continuous.

Thus, by induction, $\tilde{F}$ is defined on all $I \times I$.

Uniqueness is proven in a similar way as the previous theorem such that once $\tilde{F}$ at $(0, 0)$ is defined, the rest of the function is completely determined.

Now suppose that F is a path homotopy. The map F carries the entire left edge $0 \times I$ into a single point b_0 of B. Because $\tilde{F}$ is a lifting of F, it carries this edge into the set $p^{-1}(b_0)$. Since $0 \times I$ is connected and $\tilde{F}$ is continuous, $\tilde{F}(0 \times I)$ is connected and thus must be equal to a one-point set. Similarly, $\tilde{F}(1 \times I)$ must be a one point set. Thus $\tilde{F}$ is a path homotopy.

□

Theorem 8.3. *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1; let $\tilde{f}$ and $\tilde{g}$ be their respective lifting to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.*

Proof. Let $F : I \times I \to B$ be the path homotopy between f and g. Then $F(0, 0) = b_0$. Let $\tilde{F} : I \times I \to E$ be the lifting of F to E such that $\tilde{F}(0, 0) = e_0$. By the preceding lemma, $\tilde{F}$ is a path homotopy, so that $\tilde{F}(0 \times I) = e_0$ and $\tilde{F}(1 \times I)$ is a one-point set e_1.

The restriction of $\tilde{F}|I \times 0$ of $\tilde{F}$ to the bottom edge of $I \times I$ is a path on E beginning at e_0 that is a lifting of $F|I \times 0$. By uniqueness of path liftings, we must have $\tilde{F}(s, 0) = \tilde{f}(s)$. Similarly, $\tilde{F}|I \times 1$ is a path on E that is a lifting of $F|I \times 1$ and it begins at e_0 because $\tilde{F}(0 \times I) = e_0$. By uniqueness of path liftings, $\tilde{F}(s, 1) = \tilde{g}(s)$. Therefore, both $\tilde{f}$ and $\tilde{g}$ end at e_1, and $\tilde{F}$ is a path homotopy between them.

□

Since we know that a lifting map of a path is unique and that two homotopic paths have lifting maps that end at the same point, we can use this fact to define the lifting correspondence.

Let $p : E \to B$ be a covering map; let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$ (note that this is unique by the previous theorem). Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

We call this ϕ the lifting correspondence derived from the covering map p. It depends of course on the choice of the point e_0.

We can characterize the lifting correspondence based on whether or not E, the covering space, is path connected or simply connected.

Theorem 8.4. *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof. If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0 since $p(e_0) = p(e_1)$. Then $\phi([f]) = e_1$ by definition. Thus ϕ is surjective.

Let E be simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively, to paths in E that begin at e_0; then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g by previuos identity. Thus, $[f] = [g]$ and we have shown that ϕ is injective. Since all simply connected spaces are path connected, ϕ is also surjective. Thus, ϕ is bijective. ∎

We will now prove that the fundamental group of the unit circle is isomorphic to something very familiar.

Theorem 8.5. *The fundamental group of S^1 is isomorphic to the additive group of integers.*

Proof. Let $p : \mathbb{R} \to S^1$ be the covering map proved from earlier. Let $e_0 = 0$ and let $b_0 = p(e_0)$. Then we can see that $p^{-1}(b_0)$ is the set of integers. Since $\mathbb{R}$ is simply connected, then the lift correspondence

$$\phi : \pi_1(S^1, b_0) \to Z$$

is bijective by previous theorem. Now we must show that ϕ is a homomorphism and we have shown that it is an isomorphism. Let $[f]$ and $[g]$ be in $\pi_1(B, b_0)$. Let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on $\mathbb{R}$ beginning at 0. Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$. Then $\phi([f]) = n$ and $\phi([g]) = m$ by definition. Let $\bar{g}$ be the path

$$\bar{g} = n + \tilde{g}(s)$$

on $\mathbb{R}$. Both n and m are integers so that $p(n+x) = p(x)$ for all $x \in \mathbb{R}$. Then $\bar{g}$ is a lifting of g that begins at n. Thus, the product $\tilde{f} * \bar{g}$ is defined and it is the lifting of $f * g$ that begins at 0. We can check this since $p \circ (\tilde{f} * \bar{g}) = (p \circ \tilde{f}) * (p \circ \bar{g}) = f * g$ which shows that this is indeed a lifting map of $f * g$. The end point of this path is $\bar{g}(1) = n + m$. Then by definition,

$$\phi([f] * [g]) = \phi([f * g]) = n + m = \phi([f]) + \phi([g])$$

Thus, we have shown that it is a homomorphism and since it is bijective, it is an isomorphism. Thus, the fundamental group of the circle is isomorphic to the additive integers. □

9 Retractions and Fixed Points

To prove the Brouwer fixed-point theorem for the disk, we must first establish the definition of a retraction and prove a few theorems.

If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r|A$ is the identity map of A. If such a map r exists, we say that A is a retract of X.

Define B^2 to be the two dimensional disc as the set $\{(x,y)|x^2 + y^2 \leq 1\}$.

Theorem 9.1. *If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.*

Proof. If $r : X \to A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A. Then, $(r \circ j)_*$ is the identity map of $\pi_1(A, a)$. Then, $(r \circ j)_* = r_* \circ j_*$ so that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$. Then, j_* must be injective for $r_* \circ j_*$ to be the identity map. □

Theorem 9.2. *There is no retracton of B^2 onto S^1.*

Proof. If S^1 were a retract of B^2, then the homomorphism induced by inclusion $j : S^1 \to B^2$ would be injective. But the fundamental group of S^1 is nontrivial and the fundamental group of B^2 is trivial. This is impossible so there cannot be a retraction between the two. □

We will now prove an important lemma.

Theorem 9.3. *Let $h : S^1 \to X$ be a continuous map. Then the following conditions are equivalent: (1) h is null homotopic, (2) h extends to a continuous map $k : B^2 \to X$, and (3) h_* is the trivial homomorphism of fundamental groups.*

Proof. In order to prove this, we must show that (1) implies (2), (2) implies (3), and (3) implies (1). We will go in order.

Let h be nulhomotopic. Thus, there exists a homotopy $H : S^1 \times I \to X$ between h and the constant map. Let $\pi : S^1 \times I \to B^2$ be the map

$$\pi(x, t) = (1 - t)x.$$

We can easily see that π is continuous, closed, and surjective. It collapses $S^1 \times 1$ to the point 0. This is the only point where π is not injective. So, $\pi : S^1 \times I \to B^2 - 0$ is a bijection. By previous theorem, π^{-1} exists and is continuous. If we take the composition of H with π^{-1}, we have that $H \circ \pi^{-1} : B^2 - 0 \to X$ where $H \circ \pi^{-1}$ is continuous. If we take $(H \circ \pi^{-1})(0)$, we see that although $\pi^{-1}(0)$ is not defined, it "maps" this point to $S^1 \times 1$ which H maps to a single point. So, our map $H \circ \pi^{-1}$ exists and is continuous for $B^2 \to X$. We call this map k. k is an extension of h since if we limit the domain of k to S^1, we get the same map h as before.

We now prove (2) $\Rightarrow$ (3). Let h extend to a continuous map $k : B^2 \to X$. Let $j : S^1 \to B^2$ be the inclusion map so that h equals the composition $k \circ j$. Then, by previous theorem, $h_* = (k \circ j)_* = k_* \circ j_*$. But

$$j_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0)$$

is trivial because the fundamental group of B^2 is trivial. Therefore, h_* must be trivial.

(3) $\Rightarrow$ (1) follows similarly to (1) $\Rightarrow$ (2). Let h_* be the trivial homomorphism of fundamental groups. Let $x_0 = h(b_0)$. Consider $p : \mathbb{R} \to S^1$ being the standard covering map from before. Let $p_0 : I \to S^1$ be the restriction of p to the unit interval. We can see that $[p_0]$ generates $\pi_1(S^1, b_0)$ because p_0 lifts to a path that begins at 0 and ends at 1. Since $\pi_1(S^1, b_0)$ is isomorphic to the additive integers, $[p_0]$ corresponds to the number 1 under this isomorphism which generates the integers.

Because h_* is trivial, the loop $f = h \circ p_0$ represents the identity element of $\pi_1(X, x_0)$. Therefore, there is a path homotopy F in X between f and the constant path at x_0. Consider the map $p_0 \times i : I \times I \to S^1\times$ I where i is the identity map from I to I. This map is clearly continuous, closed, and surjective and is bijective except for the points $0 \times t$ and $1 \times t$ where they both get mapped to $b_0 \times t$ for all t. Consider the inverse map of $p_0 \times i$. It exists and is continuous by previous theorem if we ignore the points $b_0 \times t$ for all t. So the composition of F with $(p_0 \times i)^{-1}$ if we ignore those points, is continuous. We will call this map H. But, if we consider the points $b_0 \times t$ for all t, H is defined since F maps $0 \times I$ and $1 \times I$ to the point x_0 in X. Thus, $H : S^1 \times I \to X$ is continuous. We can see that H is a homotopy between h and the constant map since $(p_0 \times i)^{-1}(s, 0) = (p_0^{-1}(s), 0)$. Then $F(p_0^{-1}(s), 0) = f(p_0^{-1}(s)) = h(s)$. Likewise, $F(p_0^{-1}(s), 1)$ is equal to the constant map. Thus, H is a homotopy between h and the constant map. $\square$

Theorem 9.4. *The inclusion map $j : S^1 \to \mathbb{R}^2 - 0$ is not nulhomotopic. The identity map $i : S^1 \to S^1$ is not nulhomotopic.*

Proof. There exists a retraction of $\mathbb{R}^2 - 0$ onto S^1 by the equation $r(x) = x/\|x\|$. We can check to see that this works. Since it has a retraction, then j_* is injective by theorem. This also means that it cannot be nontrivial since S^1 has a nontrival fundamental group. Thus, it cannot be nulhomotopic. Similary, i_* is the identity homomorphism and is nontrivial by the same reasoning. □

10 Brouwer Fixed-Point Theorem for the Disc

To prove the Brouwer Fixed-Point Theorem for the Disc, we must first prove a theorem involving a vector field on B^2. After we prove this theorem, the fixed-point theorem will become a relatively easy corollary to this theorem.

Consider a vector field on B^2 by the ordered pair $(x, v(x))$, where x is in B^2 and $v(x)$ is a continuous function from B^2 to $\mathbb{R}^2$. The function $v(x)$ determines the direction that the vector field points at the given position x. We say that the vector field is nonvanishing if $v(x)$ never equals zero.

Theorem 10.1. *Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where it points directly outward.*

Proof. We will proceed by a proof by contradiction. We say that the vector field on S^1 does not point directly inward. Also, since the vector field is nonvanishing, we can say that $v(x) : B^2 \to \mathbb{R}^2 - 0$. Let w be its restriction to S^1. Because the map w extends to a map of B^2 to $\mathbb{R}^2 - 0$, it is nulhomotopic by the previous lemma.

We claim that w is homotopic to the inclusion map $j : S^1 \to \mathbb{R}^2 - 0$ by the homotopy

$$F(x, t) = tx + (1 - t)w(x)$$

for all $x \in S^1$. We need to show that this homotopy is a map from $S^1 \times I$ to $\mathbb{R}^2 - 0$ by showing it never equals zero. We can easily see that $F(x, t)$ doesn't equal zero for $t = 1$. So assume that it does equal zero somewhere. Then $tx + (1 - t)w(x) = 0$. This implies then that $w(x)$ is equal to a negative scalar multiple of x, $\frac{-t}{1-t}$, which is a contradiction since that would mean that $w(x)$ points directly inward. Thus, $F(x, t)$ is a homotopy between the two.

However, this means that a nulhomotopic map w is homotopic to a map that is not nulhomotopic, j. Thus, we derive a contradiction. Therefore, the vector field must point inward at some point. Similar arguments can be used to prove that the vector field must point outward.

□

Now we will prove the Brouwer fixed-point theorem for the disc.

Theorem 10.2. *If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.*

Proof. We proceed by contradiction. Suppose that $f(x) \neq x$ for every $x \in B^2$. Then we can define a vector field $v(x) = f(x) - x$ which is nonvanishing. But this vector field cannot point directly outward, for that would mean

$$f(x) - x = ax$$

for some positive real number. This means $f(x) = (1 + a)x$ which would lie outside of B^2. Thus, we arrive at our contradiction.

□

11 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra is a well known result. The Theorem can hardly be called a fundamental theorem of Algebra since its scope is that of the algebra before the study of abstract structures such as groups, rings, etc. The Theorem simply says that any polynomial of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where each coefficient is either a real or complex number has real or complex roots and $n \geq 1$. It is important to note that this is equivalent to saying that any polynomial of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with real or complex coefficients and $a_n \neq 0$ has at least one real or complex root. The reason being is that we can divide each term of the polynomial by a_n to get a polynomial of the former type while not changing the root of the polynomial. The proof we will present will utilize most of the important theorems and ideas previously discussed from homotopy theory to retractions.

Theorem 11.1. *A polynomial equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root.

Proof. The first step of the proof is to examine the map $f : S^1 \to S^1$ given the equation $f(z) = z^n$, where z is a complex number. In this discussion, we will use S^1 as a subset of the complex plane instead of $\mathbb{R}^2$. We will show that the induced homomorphism f_* which maps the fundamental group from S^1 to the fundamental group of S^1 is injectvie.

Let $p_0 : I \to S^1$ be the standard loop in S^1. That is,

$$p_0(s) = e^{2\pi is}$$

We know that $e^{2\pi s}$ = (cos $2\pi s$, sin $2\pi s$) and by defintion f_* maps this loop $p_0(s)$ to $f(p_0(s)) = (e^{2\pi is})^n$. To examine this further, we must look at the lift of p_0 into $\mathbb{R}^1$ by the map $\tilde{p_0} : I \to \mathbb{R}^1$ by the equation $\tilde{p_0}(s) = s$. Then we use the covering map $r : \mathbb{R} \to S^1$ which is given by $r(s) =$ (cos $2\pi s$, sin $2\pi s$). Thus, $r(\tilde{p_0}(s)) = p_0(s)$ which means that $\tilde{p_0}$ is indeed a lift.

We then use this fact to show that p_0 corresponds to the integer 1 in the standard isomorphism with S^1 and the integers. Since the map $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is the standard isomorphism between the two and the definition of $\phi([f])$ is equal to the end point of the lift of f, $\phi([p_0]) = \tilde{p_0}(1) = 1$. Thus, we have shown that p_0 corresponds to the integer 1.

Likewise, we can show that $f_*(p_0)$ corresponds to the integer n by the same technique. We merely show that the lift is $f(\tilde{p_0})(s) = ns$ and using the same covering map r, we have that $\phi([f(p_0)]) = f(\tilde{p_0})(1) = n$. Thus, f_* maps the integer 1 to the integer n. Since, 1 generates the integers, f_* maps any integer m to mn which we can use to prove that the map is injective.

Let $f_*(s) = f_*(t)$. Then, $sn = tn$. Thus, $s = t$ which is what we needed to show that the map f_* is injective. This concludes the first step of the proof.

In the second step of the proof, we will consider the map

$$g : S^1 \to \mathbb{R}^2 - 0$$

by $g(z) = z^n$. We will show that g cannot be nullhomotopic. It is easy to see that g is the same map as f from Step 1 but with the inclusion map of $j : S^1 \to \mathbb{R}^2 - 0$ so that $g = j(f)$. By previous theorem, j_* is injective because S^1 is a retract of $\mathbb{R}^2 - 0$. We then have that $g_* = (j \circ f)_* = j_* \circ f_*$ which is injective since the composition of two injective functions is injective. Furthermore, we know that g cannot be nullhomotopic since, if it was nullhomotopic, g_* would map to the identity element of the fundamental group of $\mathbb{R}^2 - 0$. The only way for that to be true and have g_* be injective would be for the domain to be trivial. Since the domain is isomorphic to the integers, g must not be nullhomotopic.

For the third part of the proof, we will prove a special case of the theorem. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where

$$|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$$

we will show that this polynomial must have a root lying in the unit ball B^2. We precede with a proof by contradiction by assuming that there does not exist a root of his polynomial in the unit ball. If this is true, then we can define a map $k : B^2 \to \mathbb{R}^2 - 0$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

This is because the polynomial was assumed to never have a root in the unit ball. Thus, it will never map into the point 0 of $\mathbb{R}^2 - 0$. We will consider the restriction of k to be $h : S^1 \to \mathbb{R}^2 - 0$. This map h extends to a map of the unit ball into $\mathbb{R}^2 - 0$ which implies by previous theorem that h is nullhomotopic. However, will define a homotopy F between h and the map g from part two. We define $F : S^1 \times I \to \mathbb{R}^2 - 0$ by the equation

$$F(z, t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$$

We can easily see that at $t = 0$, $F(z, t) = g(z)$ from part two and that when $t = 1$, $F(z, t) = h(z)$. It is also important to point out that $F(z, t)$ never equals 0 since

$$|F(z, t)| \geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_0)|$$

by the triangle inequality,

$$\geq 1 - t(|a_{n-1}z^{n-1}| + \cdots + |a_0|)$$

$$= 1 - t(|a_{n-1}| + \cdots + |a_0|) > 0$$

This tell us that $F(z, t)$ always maps to a point in $\mathbb{R}^2 - 0$ since the magnitude of a point it maps is always greater than 0. However, we have just shown that there exits a homotopy between h and g which is a contradiction since h is nulhomotopic and g cannot be nulhomotopic. Therefore, the polynomial must have a root in the unit ball.

The last part is to show that the general case reduces to the third part of our proof and that we can express the general polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

in terms coefficients whose magnitudes add up to less than 1. So, given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

let us chooose a real number $c > 0$ and let $x = cy$ so that we have

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0.$$

We can divide through by c^n to get

$$y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_1}{c^{n-1}} y + \frac{a_0}{c^n} = 0.$$

We choose c large enough so that

$$|\frac{a_{n-1}}{c}| + |\frac{a_{n-2}}{c^2}| + \cdots + |\frac{a_1}{c^{n-1}}| + |\frac{a_0}{c^n}| < 1$$

which shows us that the previous polynomial will have a root lying in the unit ball by part three. If we find a root for this polynomial, y_0, all we need to do is set $x_0 = cy_0$ to get a root x_0 of the originial polynomial equation. We have just shown that we can find a root for any polynomial of the original form which is exactly what we needed to do to prove the Fundamental Theorem of Algebra. $\square$

12 Conclusion

In this thesis, we covered the basic notions of homotopy theory and its potential uses. I believe that the power of homotopy theory is fully demonstrated in the fact that a seemingly unrelated theorem to algebraic topology involving polynomials (the fundamental theorem of algebra) can be proved using homotopy groups. During my research, I learned a great deal about how different mathematical disciplines can relate to each other and be used as tools to investigate new theorems. This shows the beauty of mathematics as generalizations of ideas and concepts can be applied to different fields that might not otherwise have anything else in common.

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