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Inelastic Collapse in One Dimensional Systems with Ordered Collisions

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Abstract

We investigate a peculiar feature of a simple model of collisions between point particles known as *inelastic collapse.* The model assumes that all particles move in one dimension at a constant velocity between collisions, and that an important ratio of velocities called the coefficient of restitution is constant for each collisions. With these assumptions, we show that it is possible for systems of particles in one dimension to collide infinitely often. We find the conditions on the initial velocities and the coefficient of restitution for inelastic collapse to occur analytically for systems with fixed collision orders and test them using *Mathematica* simulations.

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LAKE FOREST COLLEGE
Senior Thesis

Inelastic Collapse in One Dimensional Systems with Ordered Collisions

by

Brandon R. Bauerly

April 24, 2015

The report of the investigation undertaken as a
Senior Thesis, to carry two courses of credit in
the Department of Physics

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Michael T. Orr
Krebs Provost and Dean of the Faculty

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ABSTRACT

We investigate a peculiar feature of a simple model of collisions between point particles known as *inelastic collapse*. The model assumes that all particles move in one dimension at a constant velocity between collisions, and that an important ratio of velocities called the coefficient of restitution is constant for all collisions. With these assumptions, we show that it is possible for systems of particles in one dimension to collide infinitely often. We find the conditions on the initial velocities and the coefficient of restitution for inelastic collapse to occur analytically for systems with fixed collision orders and test them using *Mathematica* simulations.

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I. Introduction

In elementary school, children are taught that there are three primary phases of matter we interact with on a daily basis: solid, liquid, and gas. However, while walking on a beach we often discover that sand can possess characteristics of all three at various times. It is extremely difficult to investigate the properties of sand by modelling it as a collection of point particles all obeying classical mechanics individually. Thus researchers have been motivated to label materials composed of many independently moving pieces by the name of “granular materials .” [1] By modelling granular materials like sand, snow, rice and ball bearings as a state of matter in their own right, we can seek to discover their similarities and differences to the traditional phases of matter.

In an ideal gas, the molecules collide with each other elastically, conserving kinetic energy. In a granular gas, a crucial difference is that the particles collide with each other inelastically, losing kinetic energy with each collision. One peculiar phenomena observed in granular materials is inelastic collapse. Intuitively, its definition is “a granular material experiencing infinite collisions in a finite time.”[2] In a real granular material, particles that collide with one another frequently lose kinetic energy with each collision and as a result get increasingly close together. As a result of this inelasticity, particles form clumps in the granular material. In this thesis, I will explore three systems which experience an infinite number of collisions. The simplest system that can experience inelastic collapse is known as The Bouncing Ball – a single point particle colliding with the ground under the influence of gravity alone. It is a simple exercise in first-year physics to show that after a finite amount of time has elapsed since the ball has been dropped, it will experience an infinite number of collisions. However, it is important to note that this only occurs under a model subject to many approximations. The conclusion of the bouncing ball is only valid in the model of a Newtonian point particle in a vacuum, with a constant coefficient of restitution experiencing instantaneous collisions.

Inelastic collapse has been observed in more complicated 1 dimensional systems as well. The systems of 2 free particles and a wall, and 3 free particles have been shown to exhibit inelastic collapse for coefficient of restitutions less than some critical values by McNamara[2] , these will be studied in more detail in this thesis. Evidence of inelastic collapse has been shown to occur in systems of n free particles both with and without a stationary wall.[3,4] Thus in 1 dimension, one can obtain inelastic collapse for an arbitrary number of particles within a certain range of the coefficient of restitution and initial velocities.

One systems are interesting in their own right, but we happen to live in three dimensions. It is reasonable to ask if inelastic collapse can occur in three or more dimensions. Inelastic collapse has been shown to occur in an arbitrary number of spatial dimensions but with a catch.[5,6] While in one dimension inelastic collapse is the end point in the "lives" of the particles, in all higher dimensions it is not an end point but an event. Once higher dimensional systems of particles experience inelastic collapse, they can separate. One can find regions of no collapse, unstable collapse, and stable collapse in the parameter space.[6]

However, If we remove some of the approximations we started with, we find that the conclusion of inelastic collapse is no longer true. Goldman and his colleagues have shown that when the coefficient of restitution is allowed to be velocity dependent, which is closer to the behavior of real physical systems we no longer get inelastic collapse for any initial values of the coefficient of restitution in the three free particle system.[7] Since the three particle system is the basis for inelastic collapse in higher numbers of particles, this also implies inelastic collapse cannot occur for a realistic 1 dimensional system containing any number of particles.

Although the mathematical idealization of having an infinite number of collisions in a finite time does not actually occur for real materials, approximate phenomena can indeed occur. We certainly do see large number of collisions occurring in a very short time in shaking a box of marbles for example. Thus the study of inelastic collapse can be used to predict when granular systems will experience many collisions in a short time and clump together as a result. Knowledge of these conditions can aid in the study of other interesting granular phenomena we can experimentally investigate.

II. A Simple Model of 1D Collisions

A. The Coefficient of Restitution

We all have some intuitive idea of collisions: two objects approach each other with initial trajectories, something happens, and the objects' trajectories are changed. In mechanics, we have developed mathematical machinery to describe these interactions in detail.

We know that for all collisions that momentum must be conserved if there is no external force acting on the system; this is a fundamental law of physics. We call a collision *elastic* if kinetic energy is conserved as well. If kinetic energy is NOT conserved, we call the collision *inelastic*. A collision is called *perfectly inelastic* if the particles stick together and share a common post-collisional velocity. Perfectly inelastic collisions lose the maximal amount of kinetic energy possible while still conserving momentum

Let's study a one-dimensional system containing two free particles of masses m_1, m_2 moving with initial velocity components v_1, v_2 colliding elastically with post-collisional velocity components v_1', v_2' . In this thesis, we are mainly going to study one-dimensional systems. In this notation the *v*'s represent the components of the particle's velocity vector in this single dimension. They should not be confused with the magnitude of the particles velocity vector as the v's can be positive or negative depending on the direction of motion. To begin studying our 1-dimensional elastic system, let's look at the conservation laws.

In all collisions the total momentum of the system before the collision is the same as its total momentum after the collision.

Conservation of momentum states

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' . \quad (1)$$

Because the collisions are also elastic, the system's kinetic energy before and after the collision are the same as well.

Conservation of kinetic energy states

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2 . \quad (2)$$

Combining these equations yields the velocity relation

$$v'_2 - v'_1 = v_1 - v_2 . \quad (3)$$

This is commonly read as "The speed of separation equals the speed of approach." For elastic collisions, how quickly the particles move apart after the collision is equal to how quickly they were initially moving together.

This velocity relation only applies to elastic collisions, but we will generalize it to inelastic collisions by defining the unitless parameter ε, known as the coefficient of restitution.

$$v'_2 - v'_1 = \varepsilon(v_1 - v_2) . \quad (4)$$

We will only consider the case where $0 \le \varepsilon \le 1$, implying that the system cannot gain energy by its particles colliding (no explosions).

Let's look at special cases of ε. If ε=1 then we recover our original velocity relation for elastic collisions

$$\varepsilon = 1 \Rightarrow v'_2 - v'_1 = v_1 - v_2 . \tag{5}$$

If ε =0 then the post-collisional velocities are the same, implying the particles stick together and the maximal amount of kinetic energy is lost while still conserving momentum

$$\varepsilon = 0 \Rightarrow v'_1 = v'_2 . \tag{6}$$

For $0 < \varepsilon < 1$ we define the collision to be *inelastic*, somewhere between the extremes of elastic and perfectly inelastic.

Let's also look at the special case of one particle being a stationary wall such that $v_2 = v_2' = 0$.

$$0 = v_2 = v_2' \Rightarrow v_1' = -\varepsilon v_1 \; . \tag{7}$$

This special case captures the essence of what the coefficient of restitution is all about. When a particle collides with a stationary wall, ε is the factor by which its speed decreases.

The coefficient of restitution generalizes the ideas of elastic and perfectly inelastic collisions. It is an extremely important parameter that is instrumental in our work on inelastic collapse. In physical systems, the coefficient of restitution says something about the materials the colliding objects are made of. As an example, two metal balls colliding with each other would have a coefficient of restitution very close to one, while two clay balls would have a coefficient of restitution of almost zero. In all that follows, we will view it as a controllable parameter that determines how much energy is lost during a collision.

B. Approximations

In the rest of this thesis, we will apply the following approximations to all systems: there is only 1 spatial dimension, particles are points in vacuum, collisions are instantaneous, and the coefficient of restitution is constant for all collisions. Restricting the systems to be one-dimensional results in all collisions being head on with no rotation of the colliding particles. Particles are treated as point particles in vacuum as to not worry about any internal structure or air resistance. Collisions are regarded as being instantaneous so that only two particles can collide at the same time. This approximation becomes less effective as the time between collisions decreased. The coefficient of restitution is approximated to be constant between collisions so that we can control it. This is a good approximation over a wide range of relative velocities, but gets worse as the particles' velocities approach zero as inelastic collapse begins to occur. For almost all practical situations, these are *very* good approximations. We will show that under these assumptions that one dimensional systems can exhibit inelastic collapse.

.

III. The Bouncing Ball

Imagine a ball launched vertically upwards from the ground with initial velocity v_0 and acted upon by gravity. This system known as The Bouncing Ball will serve as our prototypical example of inelastic collapse. To understand the motivation for considering this system, let's qualitatively examine it in the elastic and inelastic cases. For elastic collisions, the ball will bounce off the ground and return to the same height after each bounce with the same time interval between collisions as shown in Fig. 1. For inelastic collisions, the ball will lose kinetic energy after each bounce and return to a lower height after each bounce with decreasing time intervals between each collision as shown in Fig.2

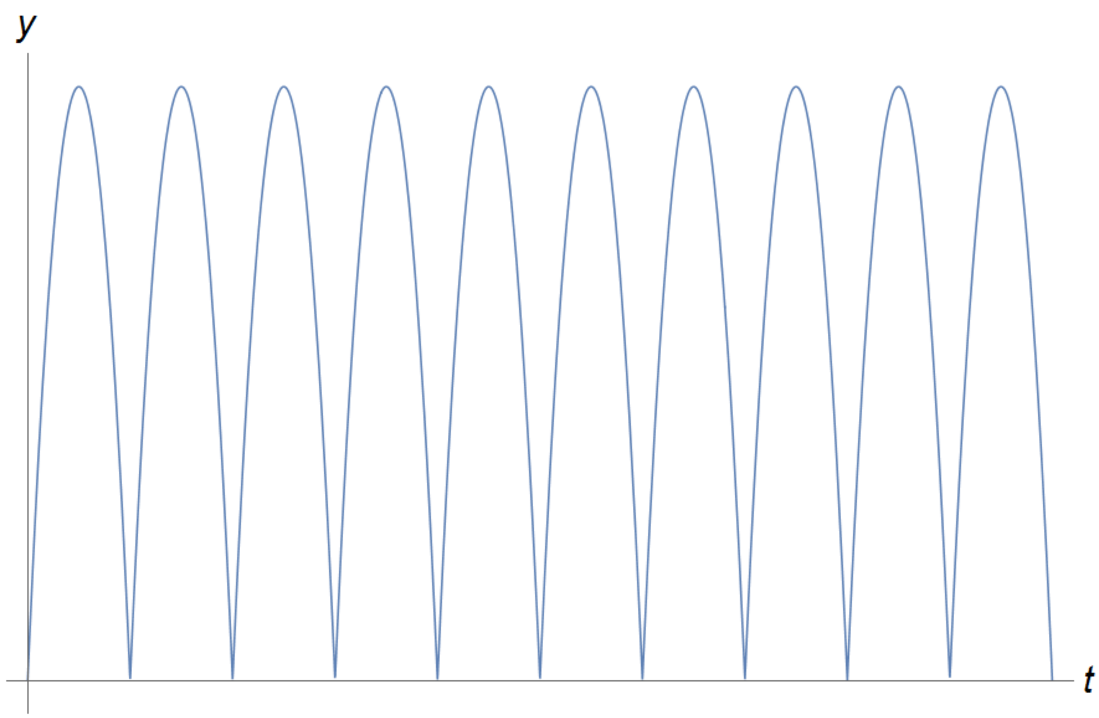

FIG.1. Vertical position vs. time for a ball launched vertically upwards from the ground and colliding elastically with the ground, $\varepsilon=1$. The ball returns to the same height after each bounce after equally spaced time intervals.

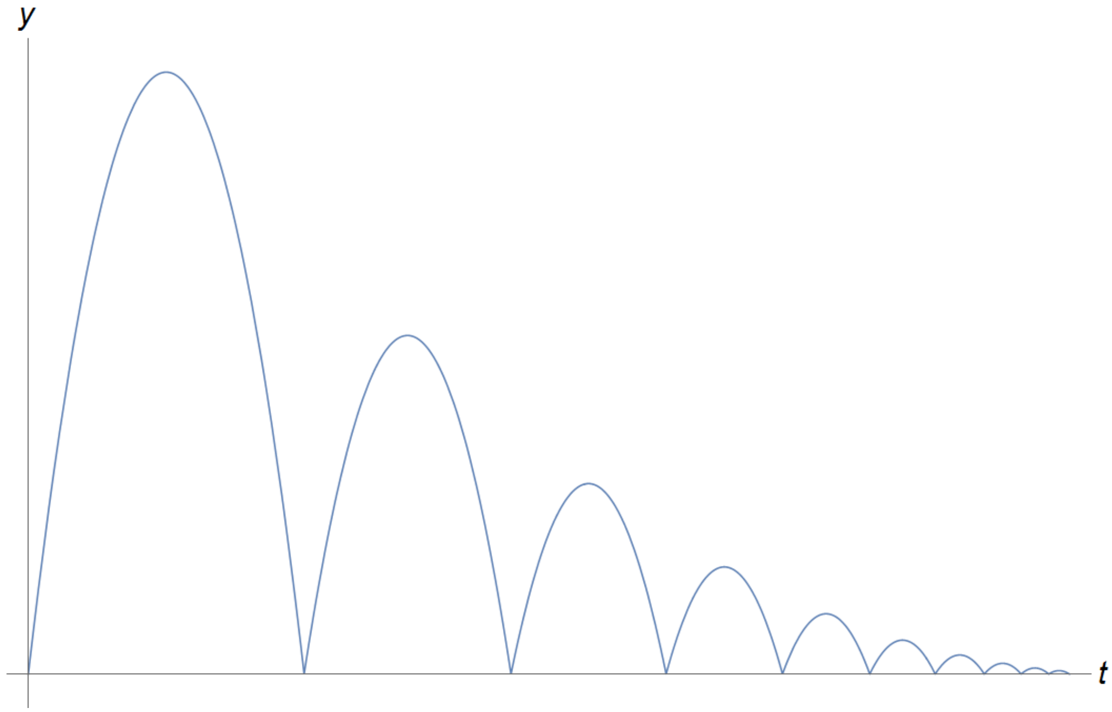

FIG. 2. Vertical position vs. time for a ball launched vertically upwards from the ground and colliding inelastically with the ground with ε=.75. The ball returns to a lower height after each bounce with decreasing time intervals.

The ball begins on the ground and is given in an initial velocity v_0 directed vertically upwards. Conservation of energy dictates it will return to the ground with a velocity v_0 directed vertically downwards. From first-year kinematics we know the ball obeys the kinematic equation of motion

$$y(t) = v_o t - \frac{1}{2} g t^2 . \tag{8}$$

We can easily solve $y(t_1) = 0$ for t_1 to obtain the time of the first collision $t_1 = \frac{2v_0}{g}$

We will find it convenient to look at the time between collisions, and define the time interval between collisions.

$$\Delta t_N = t_{N+1} - t_N . \tag{9}$$

Therefore, we have also found the time interval between the starting point and the first collision.

$$\Delta t_0 = \frac{2v_0}{g} \ . \qquad (10)$$

The ball then collides with the ground, which of course stays fixed. From the definition of the coefficient of restitution, we can compute the ball's velocity after the first bounce.

$$v' = -\varepsilon(-v_0) = \varepsilon v_0 \ . \qquad (11)$$

Now we want to see what happens after *N* collisions. We will employ the notation $v^{(N)}$ to represent to the velocity of the ball immediately after the N^{th} collision. We can apply the definition of the coefficient of restitution over and over again to find $v^{(N)}$ for all *N*.

$$\begin{aligned} &v' = -\varepsilon(-v_0) = \varepsilon v_0 \\ &v'' = -\varepsilon(-v_1) = \varepsilon(\varepsilon v_0) = \varepsilon^2 v_0 \\ &\ldots \\ &v^{(N)} = -\varepsilon(-v^{(N-1)}) = \varepsilon(\varepsilon^{N-1} v_0) = \varepsilon^N v_0 . \end{aligned} \qquad (12)$$

It is also an immediate generalization that the equation of motion is

$$y(t) = v^{(N)}(t - t_N) - \frac{1}{2} g (t - t_N)^2 , \ \ t_N \le t \le t_{N+1} \ . \qquad (13)$$

We can now solve $y(t_{N+1}) = 0$ for t_{N+1} to obtain the time intervals.

$$\Delta t_N = \frac{2v^{(N)}}{g} = \frac{2\varepsilon^N v_0}{g} = \varepsilon^N \Delta t_0 \ . \qquad (14)$$

Let's see what happens when the particle keeps colliding with the ground forever. Formally, this means we must sum the Δt_N 's. The total time is just a geometric series, and its sum can thus be computed.

$$T \equiv \sum_{N=0}^{\infty} \Delta t_N = \sum_{N=0}^{\infty} \varepsilon^N \Delta t_0 = \frac{\Delta t_0}{1-\varepsilon} < \infty \ . \tag{15}$$

The total time is *finite.* The Bouncing Ball experiences an *infinite* number of collisions in a *finite* time! While it might seem strange worded in this manner, it conforms to our experience because when we bounce a ball, we *do* indeed see it collide many times with the ground with ever decreasing time intervals between collisions. For a realistic ball, each collision does take some small time to occur which leads to the collisions overlapping and the ball coming to rest on the ground.

From the Bouncing Ball, we can formally define inelastic collapse

Definition: *Inelastic Collapse* occurs when $\sum_{N=0}^{\infty} \Delta t_N < \infty$, where $\Delta t_N = t_{N+1} - t_N$.

We have now justified our characterization of Inelastic Collapse as "infinite collisions in a finite time" by showing that for the Bouncing Ball, when an infinite number of time intervals between collisions are added up, the time in which this takes place is finite.

IV. 2 Particles and a Wall

A. The Collision Matrix Equation

Now we know that inelastic collapse can happen for The Bouncing Ball, we want to find other simple systems capable of exhibiting infinite collisions without gravity. We will examine the next simplest system in 1D that can experience inelastic collapse: 2 Particles and a Wall. Imagine a fixed wall at the origin and two equal mass free particles to the right of it.

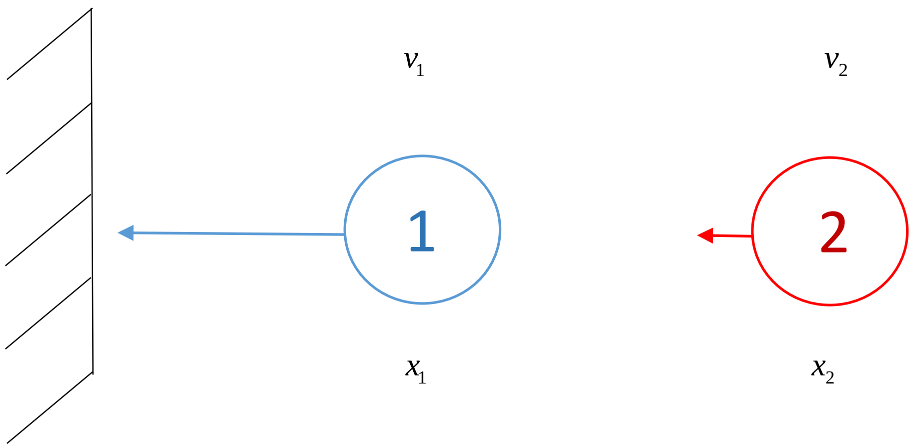

FIG. 3. A diagram of the 2 Particles and a Wall system. By definition particle 1 is between particle 2 and the wall to set the collision order. Both particles are initially heading toward the wall with particle 1's velocity the greatest.

Let the wall be at located at $x = 0$ and let the particles have initial positions x_1, x_2 where $0 < x_1 < x_2$, initial velocity components v_1, v_2 , and ε fixed. For some initial configurations there will only be a few collisions and then the particles will separate never to collide again. For other values of the initial parameters the particles will continue to collide with each other and the wall leading to an infinite number of collisions occurring. In this thesis we will study how the number of collisions *N* depends on the initial parameters but will not consider the time intervals between collisions.

To begin analyzing the system, we observe that there are two kinds of collisions in this system, and wish to know the post-collisional velocities in each case. One kind of collision occurs between particle 1 and the wall.

Particle 1 is the free particle and the wall is at rest, so we let the wall be represented by $v_{wall} = v_{wall}' = 0$ in Eq. (4) and thus obtain

$$v_1' = -\varepsilon v_1 \ . \qquad (16)$$

Since particle 2 is not participating in this collision, its post-collisional velocity is unchanged.

$$v_2' = v_2 \ . \qquad (17)$$

We see that the final velocities are linear in the initial velocities, and thus these two equations can be rewritten as a single matrix equation.

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ . \qquad (18)$$

We recast the equation in the following form to incorporate multiple collisions more conveniently.

$$\vec{v}' = c_{01}\vec{v} \quad \text{where}$$
$$\vec{v}' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}, \ c_{01} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \qquad (19)$$

It should be noted that $\vec{v}$ is a column vector and not the usual kind of velocity vector we deal with in physics. The vector $\vec{v}$ has no physically meaningful magnitude nor direction; it simply represents an ordered pair of the velocity components. It exists solely to do linear algebra on the velocities.

The other variety of collision in this system occurs between particle 1 and 2. For this case we need the definition of the coefficient of restitution and conservation of momentum for two equal mass particles.

$$\begin{aligned} v'_2 - v'_1 &= \varepsilon(v_1 - v_2), \\ v_1' + v_2' &= v_1 + v_2 . \end{aligned} \tag{20}$$

Upon solving this linear system we obtain the post collisional velocities.

$$\begin{aligned} v_1' &= \frac{1-\varepsilon}{2} v_1 + \frac{1+\varepsilon}{2} v_2, \\ v_2' &= \frac{1+\varepsilon}{2} v_1 + \frac{1-\varepsilon}{2} v_2. \end{aligned} \tag{21}$$

Once again we can rewrite this as a matrix equation.

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-\varepsilon & 1+\varepsilon \\ 1+\varepsilon & 1-\varepsilon \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} . \tag{22}$$

We can make similar definitions to the wall-particle collision case.

$$\begin{aligned} &\vec{v}^{\,'} = c_{12}\vec{v} \text{ where} \\ &\vec{v}^{\,'} = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, c_{12} = \frac{1}{2} \begin{pmatrix} 1-\varepsilon & 1+\varepsilon \\ 1+\varepsilon & 1-\varepsilon \end{pmatrix}. \end{aligned} \tag{23}$$

We want to make a further simplification to our system to reduce the number of cases we have to deal with. In all that follows we will assume that both particles are initially heading towards the wall ($v_1, v_2 < 0$) such that particle 1 collides with the wall first. This is a useful assumption to make because one of the fundamental properties of this system is that *collisions are ordered.* After the first collision, particle 1 is moving away from the wall towards particle 2. If the collisions are to continue, then particle 1 MUST collide with particle 2. After the 2nd collision particle 1 cannot collide with particle 2 again, and must collide with the wall, and the pattern repeats.

It can be easily seen that all inelastic collapse scenarios stem from particle 1 colliding with the wall first. If particle 1 and 2 collide, then either particle 1 collides with the wall next and we have our condition after the first collision, or the particles separate and inelastic collapse does not occur.

We will exploit the fact that the collisions are ordered, again and again, to create powerful mathematical tools to analyze the collisions of the 2 particles and a wall system.

Let's assume we have initial conditions suitable to get at least two collisions. We immediately know the final velocities from our matrix equations and the collision order. First, particle 1 collides with the wall, and then particle 1 collides with particle 2. We represent this by the matrix equation.

$$\begin{pmatrix} v_1'' \\ v_2'' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\varepsilon & 1+\varepsilon \\ 1+\varepsilon & 1-\varepsilon \end{pmatrix}\begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\vec{v}'' = X_2\vec{v} \text{ where} \qquad (24)$$

$$X_2 = c_{12}c_{01}.$$

Now let's assume we get at least three collisions. After particle 1 and particle 2 have collided, then the next collision must be between particle 1 and the wall.

$$\begin{pmatrix} v_1''' \\ v_2''' \end{pmatrix} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1-\varepsilon & 1+\varepsilon \\ 1+\varepsilon & 1-\varepsilon \end{pmatrix} \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\vec{v}''' = X_3 \vec{v} \text{ where} \tag{25}$$

$$X_3 = c_{01} c_{12} c_{01} .$$

For an arbitrary number of collisions *N*, we call X_N "The Collision Matrix" and define it in the velocity relation known as "The Collision Matrix Equation."

$$\vec{v}^{(\mathrm{N})} = X_N \vec{v} \ . \tag{26}$$

Our goal is to determine X_N for general *N*. We can do this by breaking it into even and odd cases. We see as an immediate generalization of our *N*=2 and 3 cases that we can rewrite the collision matrix.

$$X_N = \begin{Bmatrix} \overbrace{c_{01} \ldots c_{12} c_{01}}^{N \text{ times}} & N \text{ odd} \\ \overbrace{c_{12} \ldots c_{12} c_{01}}^{N \text{ times}} & N \text{ even} \end{Bmatrix} . \tag{27}$$

Although this is indeed a generalization that embodies the spirit of our original investigations, it is cumbersome to work with. Alternatively, we see that we can group the matrices together and raise them to the N^{th} matrix power.

$$X_N = \begin{Bmatrix} c_{01} c^{\frac{N-1}{2}} & N \text{ odd} \\ c^{\frac{N}{2}} & N \text{ even} \end{Bmatrix} \text{ where} \tag{28}$$

$$c = c_{12} c_{01} .$$

If we plug in various values of *N*, it is obvious that X_N reduces to all our previous knowledge and we can finally celebrate and write "The Collison Matrix Equation" as

$$\begin{aligned}
&\vec{v}^{(N)} = X_N \vec{v} \text{ where} \\
&\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{v}^{(N)} = \begin{pmatrix} v_1^{(N)} \\ v_2^{(N)} \end{pmatrix} \\
&X_N = \left\{ \begin{matrix} c_{01} c^{\frac{N-1}{2}} & N \text{ odd} \\ c^{\frac{N}{2}} & N \text{ even} \end{matrix} \right\} \\
&c_{01} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \; c_{12} = \frac{1}{2} \begin{pmatrix} 1-\varepsilon & 1+\varepsilon \\ 1+\varepsilon & 1-\varepsilon \end{pmatrix} \\
&c = c_{12} c_{01}.
\end{aligned} \tag{29}$$

Although it may seem as all we've done is make a lot of definitions and multiply matrices, this equation really is at the core of all our one dimensional work. The beauty and computational significance is that if we want to collide two particles and a wall *N* times, we can compute exactly what the final velocities will be afterwards. Possibly the most powerful of all uses, however, is that we can have complete control over the collisions. If we say "We want these particles to collide exactly 39 times" we can assume the particles collide at least 38 times, and find suitable velocities such that the 39^{th} collision is definitely the last one. The Collision Matrix Equation also will play an important part in computer simulations, and later we will take the limit as *N*->∞ of The Collision Matrix and derive the conditions for inelastic collapse directly.

B. The Boundary Functions

Our goal in studying this system is to find the number of collisions *N* for any initial configuration.

$$\text{Given}:\{\varepsilon, x_1, x_2, v_1, v_2\},\ \text{Find}: N\ . \tag{30}$$

The first step towards this goal is to realize that we can reduce the number of parameters we need in order to determine *N*. The initial positions x_1, x_2 simply determine the times at which collisions happen while having no effect on their order. Now we need only the coefficient of restitution and the initial velocities to determine the number of collisions. But we can simplify even further. Of our initial velocities v_1, v_2 we can choose v_1 arbitrarily to set the units and measure v_2 in units of v_1 . Therefore, we can obtain all the information about *N* by looking at the ratio $\frac{v_1}{v_2}$. We now have reduced our parameter space needed to completely describe the system from 6 variables to 2. Now we can state our simpler much simpler goal.

$$\text{Given}:\left\{\varepsilon, \frac{v_1}{v_2}\right\},\ \text{Find}: N\ \ . \tag{31}$$

We will accomplish this goal by *assuming* there are at most *N* collisions and seeing what values of our initial parameters are required to make this true.

Since both particles are initially moving towards the wall, it is impossible to get exactly 1 collision. Therefore, there must be at least 2 collisions. We will consider the cases of *N*=2 and 3 to generalize from them.

Let's say we want exactly 2 collisions. This means particle 1 collides with the wall, particle 1 and 2 collide, and then particle 1 never collides with the wall again. This condition is equivalent to $v_1" > 0$. We obtain this post-collisional velocity from The Collision Matrix Equation to be $v_1" = -\varepsilon \frac{1-\varepsilon}{2} v_1 + \frac{1+\varepsilon}{2} v_2 \geq 0$. We can easily solve for the ratio of initial velocities: $\frac{v_1}{v_2} \geq \frac{1+\varepsilon}{\varepsilon(1-\varepsilon)}$. If we define the boundary function $B_2(\varepsilon) = \frac{1+\varepsilon}{\varepsilon(1-\varepsilon)}$ then we see that B_2 divides the $\frac{v_1}{v_2}$ vs. ε parameter space into 2 regions: one with exactly 2 collisions and one with 3 or more collisions. Thus the name "Boundary Function" is motivated. At this point we know this about the parameter space: $\frac{v_1}{v_2} \geq B_2(\varepsilon) \Rightarrow N = 2$.

This time we want exactly 3 collisions and no more. This means particle 1 collides with the wall, particle 1 and 2 collide, particle 1 collides with the wall, and then particles 1 and 2 never collide again. This condition is equivalent to $v_1''' \leq v_2"$. We easily obtain these post-collisional velocities from The Collision Matrix Equation and satisfy the inequality $\frac{1}{2} v_1 (1-\varepsilon)\varepsilon^2 - \frac{1}{2} v_2 \varepsilon (1+\varepsilon) \leq \frac{1}{2} v_2 (1-\varepsilon) - \frac{1}{2} v_1 \varepsilon (1+\varepsilon)$. Upon solving for the ratio of velocities we obtain $\frac{v_1}{v_2} \geq \frac{1+\varepsilon^2}{\varepsilon + 2\varepsilon^2 - \varepsilon^3} \equiv B_3(\varepsilon)$. We easily see that $B_3(\varepsilon) < B_2(\varepsilon)$ for all ε . We already know the conditions for the *N*=2 case, therefore we exclude these points in parameter space from our analysis and obtain $B_3(\varepsilon) \le \frac{v_1}{v_2} < B_2(\varepsilon) \Rightarrow N = 3$. We can plot these curves and graphically see the conditions to get 2 or 3 collisions in Fig. 4.

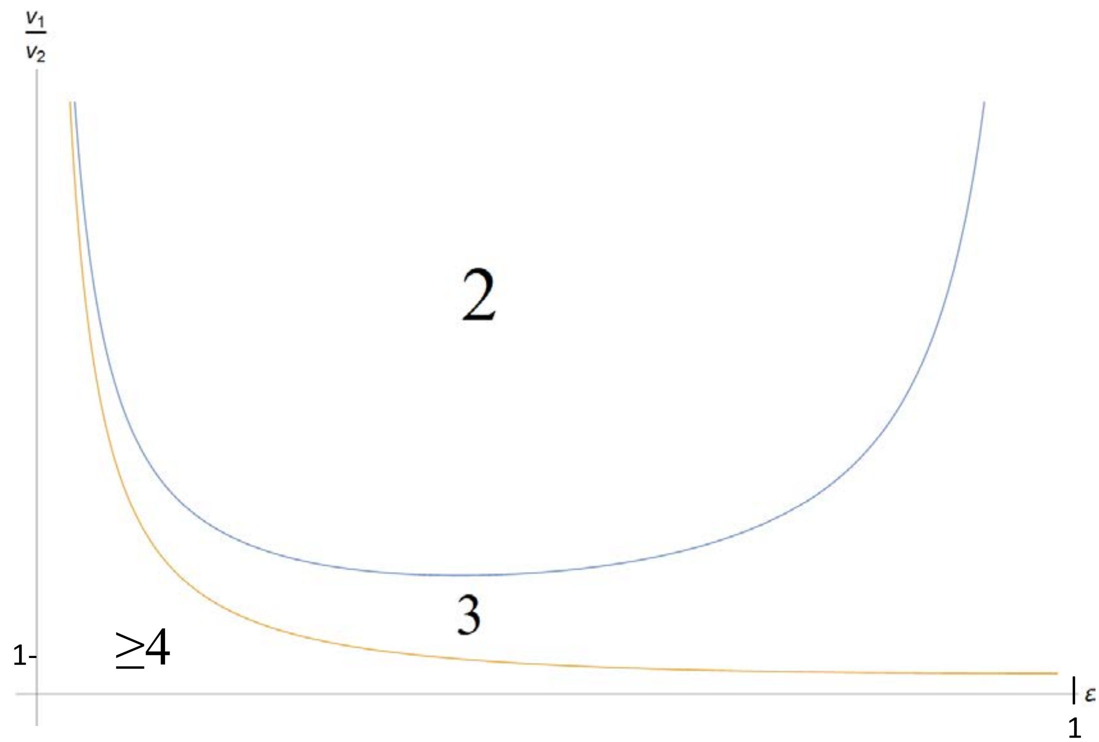

FIG. 4. A plot of the first two boundary functions $B_2(\varepsilon), B_3(\varepsilon)$. These boundary functions divide the $\frac{v_1}{v_2} vs. \varepsilon$ parameter space into regions with 2,3,and greater than 3 collisions.

From these motivational examples of 2 and 3 collisions, we can directly generalize the boundary functions to *N* collisions.

$$B_N(\varepsilon) \text{ is the solution for } \frac{v_1}{v_2} \text{ of } \begin{Bmatrix} v_1^{(N)} = v_2^{(N)} & N \text{ odd} \\ v_1^{(N)} = 0 & N \text{ even} \end{Bmatrix} . \tag{32}$$

We can also generalize their significance to the parameter space

$$B_N(\varepsilon) \le \frac{v_1}{v_2} < B_{N-1}(\varepsilon) \Leftrightarrow N \text{ collisions} . \tag{33}$$

At this point achieving our goal is reduced to the mathematical problem of computing the Boundary Functions. In this work we will use three distinct mathematical methods to compute the finite boundary functions: *Mathematica* solutions using the Collision Matrix , analytic solutions using nested functions, and diagonalizing the collision matrix.

To implement the direct solution using *Mathematica*, a program was written that, given the collision matrix and the rules for computing the final velocities, could apply the conditions in Eq. (31) to solve for the ratio of velocities for a given *N*. This method is actually *extremely* efficient in computing the finite boundary functions up to *N*=500 or so, at which point it takes approximately one minute to compute compared to a few seconds.

Through the power of the collision matrix and *Mathematica*, we can find and plot as many of the boundary functions we want and determine their exact form.

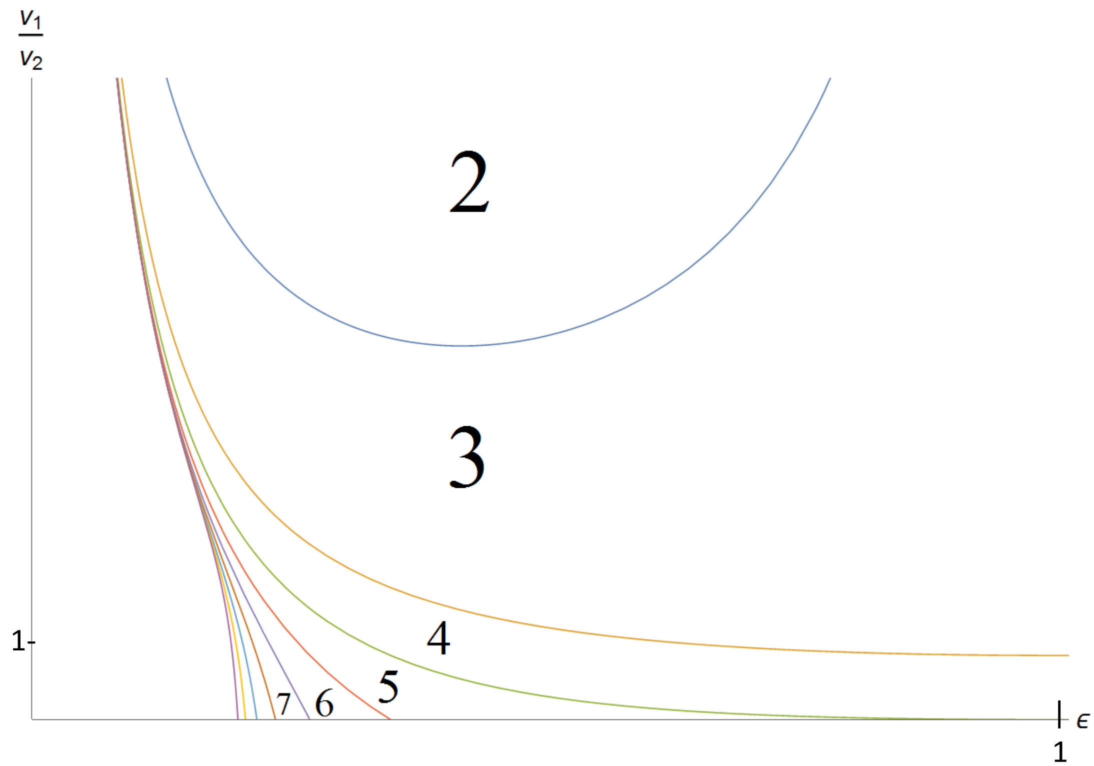

FIG.5. A plot of the first nine boundary functions $B_2(\varepsilon)-B_{10}(\varepsilon)$. These boundary functions divide the $\frac{v_1}{v_2}vs.\varepsilon$ parameter space into regions containing the indicated number of collisions.

$$
\begin{aligned}
B_2(\varepsilon) &= \frac{1+\varepsilon}{\varepsilon-\varepsilon^2} \\
B_3(\varepsilon) &= \frac{1+\varepsilon^2}{\varepsilon+2\varepsilon^2-\varepsilon^3} \\
B_4(\varepsilon) &= -\frac{(-1+\varepsilon)^2(1+\varepsilon)}{\varepsilon(-1-\varepsilon-3\varepsilon^2+\varepsilon^3)} \\
B_5(\varepsilon) &= \frac{1-2\varepsilon-2\varepsilon^2-2\varepsilon^3+\varepsilon^4}{\varepsilon+4\varepsilon^4-\varepsilon^5}
\end{aligned}
\tag{34}
$$

The Boundary Functions are especially significant for us because in the literature review, we have found no other researcher claiming to have found the conditions for an arbitrary number of finite collisions.

These parameter space graphs give us COMPLETE control over the system. We can now choose initial parameters that will give us ANY finite number of collisions.

Unfortunately, asking *Mathematica* to compute the collision matrix and apply the definition of the boundary functions for us will not precisely determine the Inelastic Collapse Region: the subset of the parameter space that will yield infinitely many collisions.

C. Dynamical Systems and Fixed Points

Now that we've achieved our original goal to find the number of collisions in terms of the initial parameters, we must analyze the information we've obtained from the Boundary Functions to find which region of parameter space will result in *infinite* collisions. Naturally, our first instinct is simply to take the limit as N->∞ of the Boundary Functions to find what we call the Inelastic Collapse Curve

$$\mathcal{B} \equiv B_{\infty}(\varepsilon)\,. \tag{35}$$

Let's write down our current definition of the Boundary functions to see how difficult of a goal this really is.

$$\begin{aligned}
&B_N(\varepsilon) \text{ is the solution for } \frac{v_1}{v_2} \text{ of } \begin{Bmatrix} v_1^{(N)} = v_2^{(N)} & N \text{ odd} \\ v_1^{(N)} = 0 & N \text{ even} \end{Bmatrix} \text{ where} \\
&\vec{v}^{(N)} = X_N \vec{v} \\
&\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{v}^{(N)} = \begin{pmatrix} v_1^{(N)} \\ v_2^{(N)} \end{pmatrix} \\
&X_N = \begin{Bmatrix} c_{01} c^{\frac{N-1}{2}} & N \text{ odd} \\ c^{\frac{N}{2}} & N \text{ even} \end{Bmatrix} \\
&c_{01} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix},\ c_{12} = \frac{1}{2}\begin{pmatrix} 1-\varepsilon & 1+\varepsilon \\ 1+\varepsilon & 1-\varepsilon \end{pmatrix} \\
&c = c_{12} c_{01}\,.
\end{aligned} \tag{36}$$

To recast the definition of the boundary functions in a more mathematically transparent form, I will employ the methods of nested functions and fixed point analysis. This method will allow me to compute both the finite boundary functions and the inelastic collapse curve. I want to begin by writing the ratio of post collisional velocities after one and two collisions in terms of the ratio of the initial velocities. To start thinking in this new way I define the ratio of velocities to be *a*.

$$\begin{aligned} a &\equiv \frac{v_1}{v_2} \\ a^{(N)} &\equiv \frac{v_1^{(N)}}{v_2^{(N)}} \end{aligned} \quad . \tag{37}$$

It can be easily shown by computing the post-collisional velocities for one and two collisions and taking their ratio that they can be rewritten as

$$\begin{aligned} a'(a) &= -\varepsilon a \\ a''(a) &= \frac{1+\varepsilon-\varepsilon(1-\varepsilon)a}{1-\varepsilon-\varepsilon(1+\varepsilon)a} \end{aligned} \quad . \tag{38}$$

The meaning of these equations is that given an initial ratio of velocities, these equations tell us what happens after one and two collisions. Their usefulness is that we can nest them to see what happens after *N* collisions.

To more easily nest them, lets give new names to *a'(a)* and *a''(a).*

$$\begin{aligned} f(a) &\equiv a'(a) = -\varepsilon a \\ g(a) &\equiv a''(a) = \frac{1+\varepsilon-\varepsilon(1-\varepsilon)a}{1-\varepsilon-\varepsilon(1+\varepsilon)a} \end{aligned} \quad . \tag{39}$$

Since *f* goes from the initial configuration to the first collision and *g* goes from the initial configuration to the second collision, it is clear that we can write the ratio of velocities after the N^{th} collision as

$$a^{(N)} = \begin{cases} f\left(g^{\frac{N-1}{2}}(a)\right) & N \text{ odd} \\ g^{\frac{N}{2}}(a) & N \text{ even} \end{cases} . \tag{40}$$

Here the superscript notation means to nest the function with itself the indicated number of times.

The Boundary Functions can also be written in terms of *a*, *f*, and *g*.

$$B_N(\varepsilon) \text{ is the solution for } a \text{ of } \begin{cases} a^{(N)} = 1 & N \text{ odd} \\ a^{(N)} = 0 & N \text{ even} \end{cases} . \tag{41}$$

At this point the new boundary functions are not that useful. *Mathematica* actually takes significantly more time to nest these functions together than it did to multiply matrices. Computing the boundary functions for a given *N* still requires *Mathematica* to nest them, and thus no analytic information can be obtained from this new definition. Their usefulness is how we can use this new method of nesting functions that goes from one collision to the next to find the inelastic collapse curve.

Our function *g(a)* takes the initial *a* two collisions into the future. By the property of ordered collisions, each time we apply *g(a)* the same two particles are about to collide again. By applying *g* to *a* forever we make the particles collide infinitely often. Situations like this are studied in depth using Dynamical Systems Theory.

To find the inelastic collapse curve, we will borrow a technique from dynamical systems theory: fixed point analysis. We seek to "take the limit" as *N*->∞ of the boundary functions by nesting the function *g(a)* with itself infinitely many times. To do this, let's define what is meant by a fixed point.

Definition: x_0 is a "fixed point" of *f(x)* $\Leftrightarrow$ $f(x_0) = x_0$. (42)

Conceptually, the fixed points are important because if the system begins at x_0, applying *f* to it will not change its value.

The significance of fixed points for us is that now we can find the limiting behavior of the boundary functions without taking any limits or nesting any functions but simply solving the equation *g(a)=a* for a and choosing a particular solution. It can easily be shown that

$$\text{The fixed points of } g(a) \text{ are } \frac{1-\varepsilon \pm \sqrt{1-6\varepsilon+\varepsilon^2}}{2\varepsilon} \; . \tag{43}$$

We see that for the fixed points to be real, the radicand must be positive implying $1-6\varepsilon+\varepsilon^2 \geq 0$. This gives us a condition on ε. We'll define it now to be the "critical value of the coefficient of restitution" and see its significance later.

$$\varepsilon \leq \varepsilon_c \equiv 3-2\sqrt{2} \approx .17 \tag{44}$$

Let's look at a graph of *g(a)* plotted with the identity map for values above and below the critical value of ε to see the graphical significance of the fixed points.

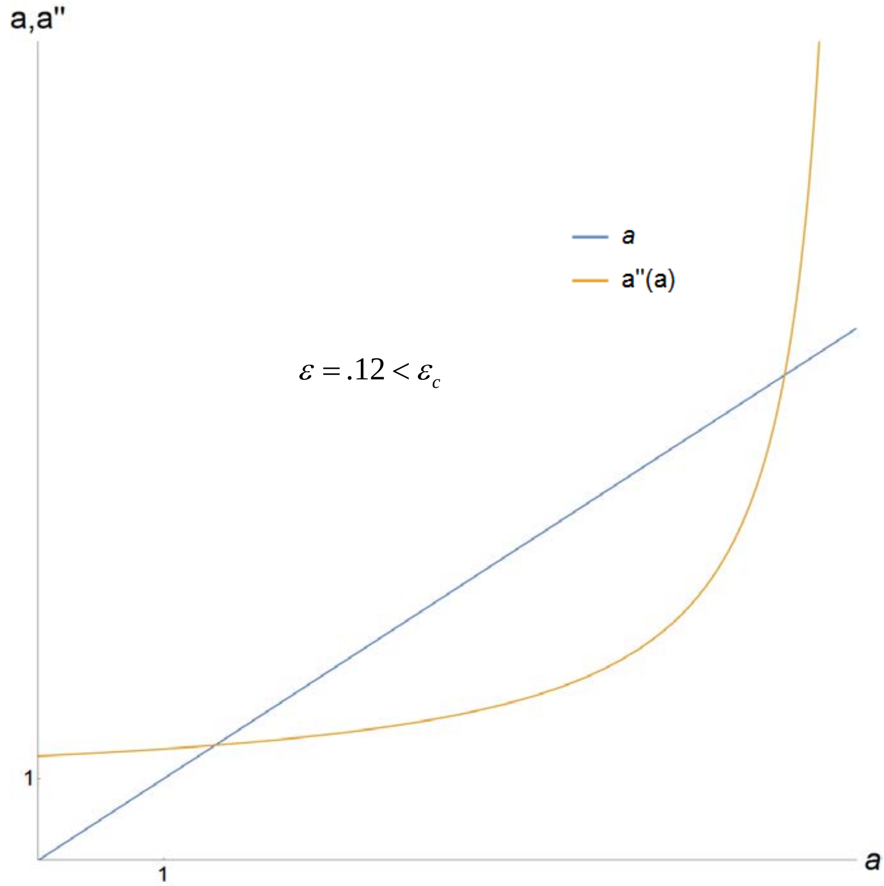

FIG. 6. Graph of the ratio of velocities after 2 collisions as a function of the initial ratio for ε=.12, below the critical value. There are 2 fixed points. The higher is unstable while the lower stable. Initial velocity ratios less than the higher fixed point will lead to inelastic collapse. Here the blue curve is the identity map and the yellow curve is the velocity ratio after two collisions as a function of the initial ratio.

In Fig. 6 we plot *g(a)* vs. *a* and compare it to the straight line identity map. The points of intersection between *g(a)* and *a* are the fixed points. Analyzing the fixed point of *g* tells us that the higher fixed point is unstable and the lower stable. For values of *a* less than the higher fixed point applying, *g* repeatedly keeps the system in this region and never satisfies the stopping conditions in the boundary functions. Therefore all values of *a* below the higher fixed point will result in an infinite number of collisions. We can now identify the Inelastic Collapse Curve as precisely this upper fixed point.

$$\mathcal{B}(\varepsilon) \text{ is the greater solution for } a \text{ of } g(a) = a . \qquad (45)$$

We simply choose the fixed point + sign before the square root that we have already obtained.

$$\mathcal{B}(\varepsilon) = \frac{1-\varepsilon+\sqrt{1-6\varepsilon+\varepsilon^2}}{2\varepsilon} \ . \qquad (46)$$

Obviously, for this to be real, $1-6\varepsilon+\varepsilon^2 \geq 0.$ This gives us a condition on the coefficient of restitution for inelastic collapse to occur, Eq. (44). The significance of this value is Inelastic collapse CAN occur for $\varepsilon \leq \varepsilon_c$, given suitable initial velocities, but CANNOT occur for $\varepsilon > \varepsilon_c$.

Let's look at the system for a value of the coefficient of restitution above the critical value as in Fig. 7.

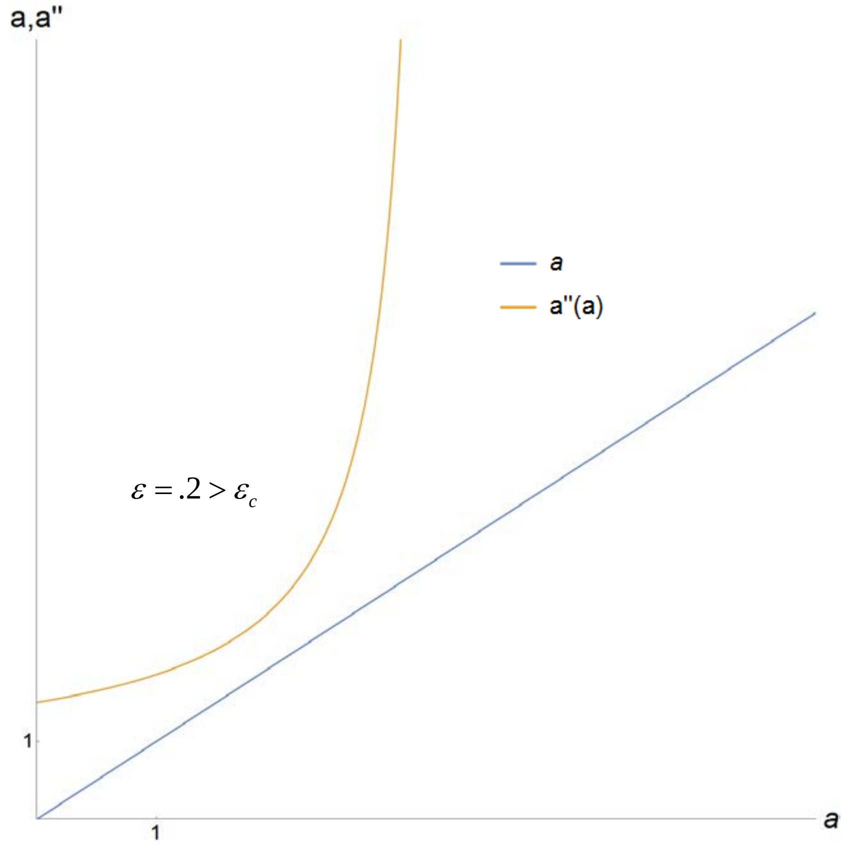

FIG. 7. Graph of the ratio of velocities after 2 collisions as a function of the initial ratio for ε=.2, above the critical value. There are no fixed points. No initial velocity ratios will lead to inelastic collapse. Here the blue curve is the identity map and the yellow curve is the velocity ratio after two collisions as a function of the initial ratio.

For a value of ε higher than the critical value there are no fixed points. This means that inelastic collapse cannot be obtained no matter what the initial velocities are.

Both viewpoints of looking at the inelastic collapse curve in parameter space and looking at the fixed points of the two collision function *g(a)* lead to the same conditions for inelastic collapse to occur. It should be noted that as written, finding the Inelastic Collapse Curve as a fixed point of *g(a)* is not completely rigorous. At this level of description it is merely a plausibility argument, and also the fastest way to obtain it. For a more rigorous though tedious approach in the next section we will diagonalize the collision matrix and take direct limits as the number of collisions grows without bound.

We have found the Inelastic Collapse Region! We now know exactly which region of parameter space will lead to the particles colliding infinitely many times. Specifically, it is written as

$$\varepsilon < 3 - 2\sqrt{2} \text{ and } \frac{v_1}{v_2} \le \mathcal{B}(\varepsilon) \Leftrightarrow \infty \text{ Collisions} . \tag{47}$$

Perhaps the best way to understand what this means is to see it illustrated graphically.

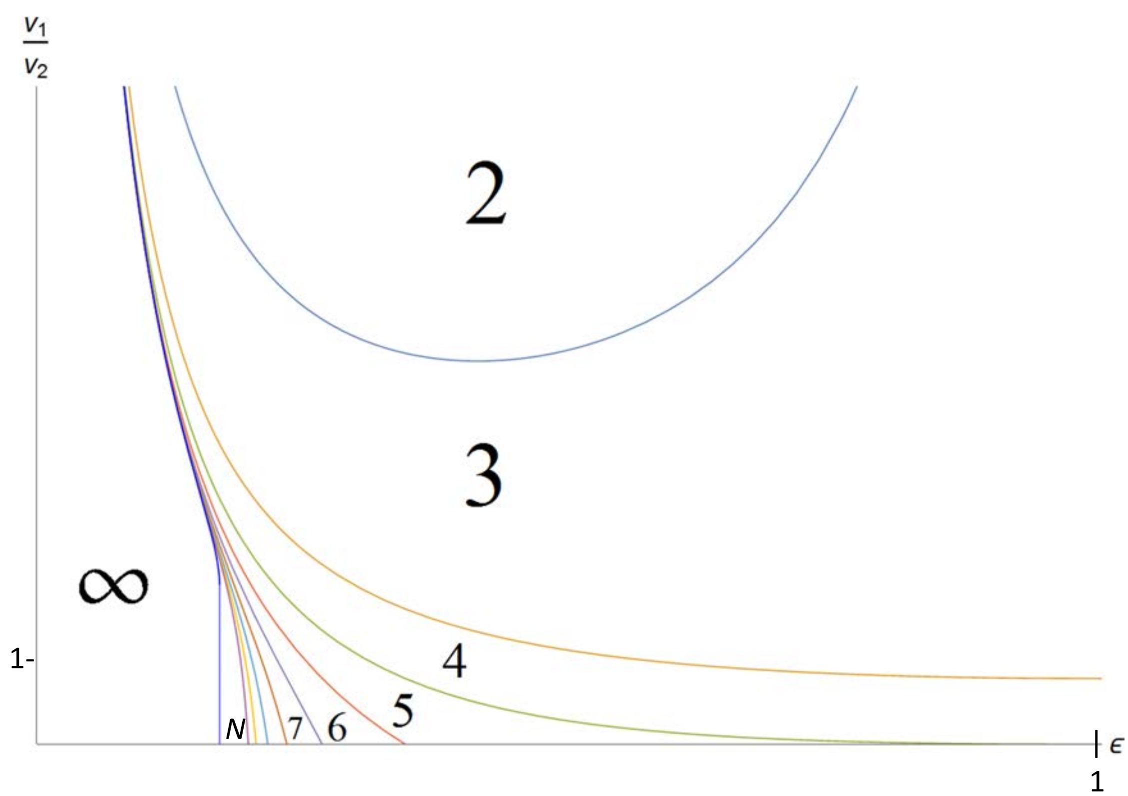

FIG.8. The completed parameter space of $\frac{v_1}{v_2}$ $vs. \varepsilon$ for the 2 Particles and a Wall system. The finite Boundary Functions can be drawn indefinitely approaching the Inelastic Collapse Curve. The region labeled ∞ is the Inelastic Collapse Region, the region where the system experiences infinite collisions.

We have already completely determined (in principle) the parameter space for finite collisions, and with the addition of $\mathcal{B}(\epsilon)$ on the far left we have mastered the infinite as well. What it means on this graph is that if we have $\varepsilon \leq \varepsilon_c = 3 - 2\sqrt{2}$ and input initial velocities within the far left blue curve, inelastic collapse will occur and the particles will keep colliding indefinitely. It should be noted that we have not actually shown that these infinite collisions occur in a finite time. This task has proven most difficult since the time intervals depend on the positions of each collision, which are non-linear in the velocities. Summing all the time intervals could prove to be a topic of future research.

D. Diagonalizing the Collision Matrix

Currently our two mathematical methods of the 2 particles and a wall system, the Collision Matrix and Dynamical Systems formulations can both compute the boundary functions for any given *N* computationally, and the dynamical systems approach can calculate the inelastic collapse curve analytically. However, at this point the analytic form of the finite boundary functions is unknown. We would like to write them down as the rational functions of *N* and ε that we know they are. A way we've found to do this is going back to our original Collision Matrix and diagonalizing it using a rudimentary linear algebra technique to put all the *N* dependencies in the eigenvalues. As an added bonus, through this method it is actually possible to take the limit as N->∞ of the boundary functions and directly compute the inelastic collapse curve by definition. Directly computing the limit is a more rigorous derivation of the Inelastic Collapse Curve than our dynamical systems analysis.

Referring to Eq. (29) we see that the only matrix that is raised to the N^{th} power is the even collision matrix $c = c_{12}c_{01}$. Thus it suffices to only diagonalize c.

The eigensystem of c can be found through standard techniques to be

$$\begin{aligned}
\lambda_1 &= \frac{1}{4}\left((1-\varepsilon)^2 - (1+\varepsilon)\sqrt{1-6\varepsilon+\varepsilon^2}\right) \\
\overline{v_1} &= \begin{pmatrix} \frac{1}{2\varepsilon}\left(1-\varepsilon+\sqrt{1-6\varepsilon+\varepsilon^2}\right) \\ 1 \end{pmatrix} \equiv \begin{pmatrix} f_1(\varepsilon) \\ 1 \end{pmatrix} \\
\lambda_2 &= \frac{1}{4}\left((1-\varepsilon)^2 + (1+\varepsilon)\sqrt{1-6\varepsilon+\varepsilon^2}\right) \\
\overline{v_2} &= \begin{pmatrix} \frac{1}{2\varepsilon}\left(1-\varepsilon-\sqrt{1-6\varepsilon+\varepsilon^2}\right) \\ 1 \end{pmatrix} \equiv \begin{pmatrix} f_2(\varepsilon) \\ 1 \end{pmatrix} .
\end{aligned} \tag{48}$$

We know c can be diagonalized and raised to the N^{th} power as

$$\begin{aligned}
c^N &= VD^NV^{-1} \ \ where \\
V &= \begin{pmatrix} f_1 & f_2 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} .
\end{aligned} \tag{49}$$

After some algebra we can simply write down the diagonalized even collision matrix.

$$c^N = \frac{1}{f_1 - f_2}\begin{pmatrix} f_1\lambda_1{}^N - f_2\lambda_2{}^N & f_1 f_2\left(\lambda_2{}^N - \lambda_1{}^N\right) \\ \lambda_1{}^N - \lambda_2{}^N & f_1\lambda_2{}^N - f_2\lambda_1{}^N \end{pmatrix}. \tag{50}$$

At this point we have to get the even and odd boundary *N* functions $B_{2N}(\epsilon)$ and $B_{2N+1}(\epsilon)$ separately according to their current definition. The goal is to compute limits as *N*->∞ of both and hope that they're the same. Let's find the even N boundary functions first.

N Even:

$$\vec{v}^{(2N)} = X_{2N}\vec{v}$$
$$\begin{pmatrix} v_1^{(2N)} \\ v_2^{(2N)} \end{pmatrix} = c^N \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} v_1^{(2N)} \\ v_2^{(2N)} \end{pmatrix} = \frac{1}{f_1 - f_2}\begin{pmatrix} \left(f_1\lambda_1^N - f_2\lambda_2^N\right)v_1 + f_1 f_2\left(\lambda_2^N - \lambda_1^N\right)v_2 \\ \left(\lambda_1^N - \lambda_2^N\right)v_1 + \left(f_1\lambda_2^N - f_2\lambda_1^N\right)v_2 \end{pmatrix}. \tag{51}$$

From this information we can solve $v_1^{(2N)} = 0$ for $\frac{v_1}{v_2}$

$$v_1^{(2N)} = \frac{1}{f_1 - f_2}\left(\left(f_1\lambda_1^N - f_2\lambda_2^N\right)v_1 + f_1 f_2\left(\lambda_2^N - \lambda_1^N\right)v_2\right) \overset{set}{=} 0$$
$$B_{2N}(\varepsilon) = \frac{f_1 f_2\left(\lambda_1^N - \lambda_2^N\right)}{f_1\lambda_1^N - f_2\lambda_2^N}. \tag{52}$$

Now we have a closed form for the even boundary functions! Not only is it very compact and beautiful, but we can actually get the inelastic collapse curve $\mathcal{B}(\epsilon)$ from this formula alone by taking the limit as *N* goes to infinity. To aid us in this quest, let's note that the functional form of λ_1 and λ_1 implies $\lambda_1<\lambda_2$ for all ε. This fact prompts us to divide numerator and denominator by λ_2^N and note that the ratio $\left(\frac{\lambda_1}{\lambda_2}\right)^N \to 0$ as *N*->∞. We can rewrite the even boundary functions as

$$B_{2N}(\varepsilon) = \frac{f_1 f_2\left(\left(\frac{\lambda_1}{\lambda_2}\right)^N - 1\right)}{f_1\left(\frac{\lambda_1}{\lambda_2}\right)^N - f_2}. \tag{53}$$

taking the limit as *N*->∞ we obtain

$$\lim_{N\to\infty} B_{2N}(\varepsilon) = \frac{1}{2\varepsilon}\left(1-\varepsilon+\sqrt{1-6\varepsilon+\varepsilon^2}\right) . \tag{54}$$

We still have to check if B_{2N+1}converges to this function as well, but if it does then we will know that inelastic collapse WILL occur for all $\frac{v_1}{v_2} \le \mathcal{B}$ for a given ϵ.

N odd:

$$\begin{aligned}
&\vec{v}^{(2N+1)} = X_{2N+1}\vec{v} \\
&\begin{pmatrix} v_1^{(2N+1)} \\ v_2^{(2N+1)} \end{pmatrix} = c_{01}c^N \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&\begin{pmatrix} v_1^{(2N+1)} \\ v_2^{(2N+1)} \end{pmatrix} = \frac{1}{f_1 - f_2}\begin{pmatrix} -\varepsilon\left[f_1\lambda_1^N - f_2\lambda_2^N \right] v_1 & -\varepsilon\left[f_1 f_2\left(\lambda_2^N - \lambda_1^N\right)\right] v_2 \\ (\lambda_1^N - \lambda_2^N)v_1 + & (f_1\lambda_2^N - f_2\lambda_1^N)v_2 \end{pmatrix}.
\end{aligned} \tag{55}$$

From this information we can solve $v_1^{(2N+1)} = v_2^{(2N+1)}$ for $\frac{v_1}{v_2}$

$$\begin{aligned}
&-\varepsilon\left[f_1\lambda_1^N - f_2\lambda_2^N \right] v_1 \quad -\varepsilon\left[f_1 f_2\left(\lambda_2^N - \lambda_1^N\right)\right] v_2 \overset{set}{=} \left(\lambda_1^N - \lambda_2^N\right) v_1 + \quad \left(f_1\lambda_2^N - f_2\lambda_1^N\right) v_2 \\
&B_{2N+1}(\varepsilon) = \frac{\lambda_2^N f_1(1+\varepsilon f_2) - \lambda_1^N f_2(1+\varepsilon f_1)}{\lambda_2^N(1+\varepsilon f_2) - \lambda_1^N(1+\varepsilon f_1)}.
\end{aligned} \tag{56}$$

We can once again divide numerator and denominator by λ_2^N and rewrite the odd boundary functions as

$$B_{2N+1}(\varepsilon) = \frac{f_1(1+\varepsilon f_2) - \left(\frac{\lambda_1}{\lambda_2}\right)^N f_2(1+\varepsilon f_1)}{(1+\varepsilon f_2) - \left(\frac{\lambda_1}{\lambda_2}\right)^N (1+\varepsilon f_1)} . \tag{57}$$

taking the limit as *N*->∞ we obtain

$$\lim_{N\to\infty} B_{2N+1}(\varepsilon) = \frac{1}{2\varepsilon}\left(1-\varepsilon+\sqrt{1-6\varepsilon+\varepsilon^2}\right). \quad (58)$$

We see that both $B_{2N}(\epsilon)$ and $B_{2N+1}(\epsilon)$ converge to the same function as N->∞. Therefore we are justified in writing the Inelastic Collapse Curve as

$$\mathcal{B}(\epsilon) = \frac{1}{2\varepsilon}\left(1-\varepsilon+\sqrt{1-6\varepsilon+\varepsilon^2}\right). \quad (59)$$

The significance of the Inelastic Collapse Curve is

$$\frac{v_1}{v_2} \le \mathcal{B}(\varepsilon) \Rightarrow \infty\, Collisions. \quad (60)$$

This condition also gives us the critical coefficient of restitution such that we get inelastic collapse, ε_c . To have real values, the quantity under the square root must be greater than or equal to zero. Applying the quadratic formula to the radicand yields the same critical epsilon that we derived in the fixed point analysis.

$$\varepsilon \le \varepsilon_c = 3-2\sqrt{2}. \quad (61)$$

But wait! There's more. In addition to computing the Inelastic Collapse Curve by definition, the eigenvalue analysis can actually be used to write the finite boundary functions in a closed form as rational functions of *N* and ε. Let's take a closer look at the boundary functions involving the eigenvalues.

$$
B_N(\varepsilon)=\begin{cases}\dfrac{\lambda_2^{\frac{N-1}{2}}\left(1+f_1\right)-\lambda_1^{\frac{N-1}{2}}\left(1+\varepsilon f_2\right)}{\lambda_2^{\frac{N-1}{2}}\left(1+\varepsilon f_2\right)-\lambda_1^{\frac{N-1}{2}}\left(1+\varepsilon f_1\right)} & N\text{ odd}\\[2ex] \dfrac{1}{\varepsilon}\dfrac{\lambda_1^{\frac{N}{2}}-\lambda_2^{\frac{N}{2}}}{f_1\lambda_1^{\frac{N}{2}}-f_2\lambda_2^{\frac{N}{2}}} & N\text{ even}\end{cases}
$$

$$
\begin{aligned}
&\textit{where}:\\
&\lambda_1=\frac{1}{4}\left(\left(1-\varepsilon\right)^2-\left(1+\varepsilon\right)\sqrt{1-6\varepsilon+\varepsilon^2}\right)\\
&f_1=\frac{1}{2\varepsilon}\left(1-\varepsilon+\sqrt{1-6\varepsilon+\varepsilon^2}\right)\\
&\lambda_2=\frac{1}{4}\left(\left(1-\varepsilon\right)^2+\left(1+\varepsilon\right)\sqrt{1-6\varepsilon+\varepsilon^2}\right)\\
&f_2=\frac{1}{2\varepsilon}\left(1-\varepsilon-\sqrt{1-6\varepsilon+\varepsilon^2}\right).
\end{aligned}
\qquad (62)
$$

Looking at the boundary functions we see that all the *N* dependencies are contained within the eigenvalues raised to some power. This motivates us to expand each *N*-dependent term using the Binomial Expansion and simplify it. By rewriting the numerator and denominator in terms of sums and differences of the eigenvalues raised to the N^{th} power and cancelling square roots, it can be show that the finite boundary functions can be rewritten as rational functions of epsilon.

$$B_N(\varepsilon)=\begin{cases}\dfrac{\sum_{k=0}^{(N-1)/2}\left(\left(1+\dfrac{\alpha}{2\varepsilon}\right)\dbinom{(N-1)/2}{2k+1}\alpha^{2((N-1)/2-2k-1)}\beta+\dfrac{1}{2\varepsilon}\dbinom{(N-1)/2}{2k}\alpha^{2((N-1)/2-2k)}\right)(\beta\gamma)^{2k}}{\sum_{k=0}^{(N-1)/2}\left(\left(1+\dfrac{\alpha}{2}\right)\dbinom{(N-1)/2}{2k+1}\alpha^{2((N-1)/2-2k-1)}\beta-\dfrac{1}{2}\dbinom{(N-1)/2}{2k}\alpha^{2((N-1)/2-2k)}\right)(\beta\gamma)^{2k}} & N \text{ odd}\\[2ex] \dfrac{2\sum_{k=0}^{N/2}\dbinom{N/2}{2k+1}\alpha^{2(N/2-2k-1)}\beta^{2k+1}\gamma^{2k}}{\sum_{k=0}^{N/2}\left(\dbinom{N/2}{2k+1}\alpha^{2(N/2-2k)-1}\beta-\dbinom{N/2}{2k}\alpha^{2(N/2-2k)}\right)(\beta\gamma)^{2k}} & N \text{ even}\end{cases}$$

where

$$\alpha=1-\varepsilon$$
$$\beta=1+\varepsilon \qquad (63)$$
$$\gamma=\sqrt{1-6\varepsilon+\varepsilon^2}\,.$$

This confirms their form as rational functions of epsilon as all the square roots either cancel out or are raised to an even power to become polynomials.

We've seen how our three mathematical techniques of using the collision matrix and *Mathematica*, dynamical systems and fixed points, and eigenvalue analysis and limits have all completely determined the $\frac{v_1}{v_2}$ vs. ε parameter space. The key fact in each model is that the collisions are completely ordered. For all intents and purposes, the 2 particle and a wall system has been solved exactly.

V. 3 Free Particles

A. The Collision Matrix Equation

We now know of two systems capable of undergoing inelastic collapse: the Bouncing Ball and 2 Particles and a Wall. Now let's consider a very similar 1-dimensional system in which almost all our results carry over: 3 Free Particles. Let there be 3 free particles in vacuum with initial positions $x_1 < x_2 < x_3$, initial velocity components v_1, v_2, v_3 and ε fixed.

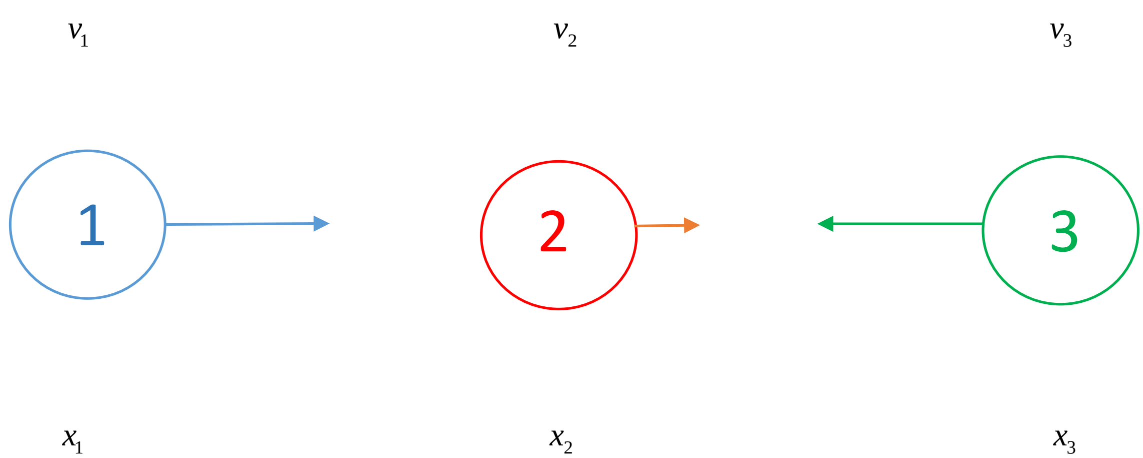

FIG.9. A diagram of the 3 Free Particles system. By definition particle 2 is between particle 1 and 3 to set the collision order.

Just like in the 2 particles and a wall system, we observe that there are two kinds of collisions in this system, and wish to know the post-collisional velocities in each case. Either particle 1 collides with particle 2 or particle 2 collides with particle 3; there is no wall to collide with. We already worked out the collision matrices for particle-particle collisions in the 2 particles and a wall system, therefore all we have to do now is rewrite everything in terms of 3x3 matrices.

If particle 1 collides with particle 2 we simply need to augment Eq. (21) with an extra velocity relation corresponding to particle 3 not participating in the collision. We then have

$$\begin{aligned} v_1' &= \frac{1-\varepsilon}{2} v_1 + \frac{1+\varepsilon}{2} v_2 \\ v_2' &= \frac{1+\varepsilon}{2} v_1 + \frac{1-\varepsilon}{2} v_2 \\ v_3' &= v_3 \end{aligned} \quad . \tag{64}$$

We can of course write this as an equivalent matrix equation

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} & 0 \\ \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} . \tag{65}$$

If particle 2 collides with particle 3 the velocity relation is very similar, except that now particle 1 is not participating in the collision

$$\begin{aligned} v_1' &= v_1 \\ v_2' &= \frac{1-\varepsilon}{2} v_2 + \frac{1+\varepsilon}{2} v_3 \\ v_3' &= \frac{1+\varepsilon}{2} v_2 + \frac{1-\varepsilon}{2} v_3 \end{aligned} \quad . \tag{66}$$

Writing this as an equivalent matrix equation we obtain

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} \\ 0 & \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} . \tag{67}$$

We are also going to make an assumption about the collision order similar to the 2 particles and a wall system. In our 3 particle system we will impose the condition that $v_1 > v_2$ and $v_2 > v_3$ such that particle 1 collides with particle 2 first. We make this assumption because all other initial conditions either lead to a small number of collisions or to a system of this form after a small number of collisions.

Once again, the fact the collisions are ordered makes this assumption very useful. The order of collisions is: particle 1 collides with particle 2, particle 2 collides with particle 3, repeat.

Through a process exactly analogous to our earlier system, we can create a Collision Matrix Equation to find all the post collisional velocities. By following the *exact same* logic we previously employed we can simply write down the Collision Matrix Equation for 3 Free Particles as a 3x3 matrix equation.

$$
\begin{aligned}
&\vec{v}^{(N)} = X_N \vec{v} \text{ where} \\
&\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \vec{v}^{(N)} = \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \\
&X_N = \begin{Bmatrix} c_{12} c^{\frac{N-1}{2}} & N \text{ odd} \\ c^{\frac{N}{2}} & N \text{ even} \end{Bmatrix} \\
&c_{12} = \begin{pmatrix} \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} & 0 \\ \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, c_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} \\ 0 & \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \end{pmatrix} \\
&c = c_{23} c_{12}.
\end{aligned}
\tag{68}
$$

B. The Boundary Functions

Our goal in studying this system is still to find the number of collisions *N* for any initial configuration.

$$\text{Given}:\{\varepsilon, x_1, x_2, \mathrm{x}_3, v_1, v_2, v_3\},\ \text{Find}: N \tag{69}$$

We can again simplify the system by reducing the number of parameters needed to determine *N*. The positions still just determine the time scale, and are not needed to determine *N*. We choose one velocity to set the units, another to set the reference frame of the middle particle being at rest, and the other is free parameter. Therefore, the number of parameters needed to describe the system is reduced from 8 to 2. We choose to combine the velocities into a new parameter alpha.

$$\alpha \equiv \frac{v_3 - v_2}{v_1 - v_2} \ . \tag{70}$$

In the reference frame where the middle particle is initially at rest, this new parameter reduces to the velocity ratio of the other two particles. We chose to define α with v_3 in the numerator so that the parameter space looks similar to the 2 particle and a wall system.

Our final goal can now be stated.

$$\text{Given}:\{\varepsilon, \alpha\},\ \text{Find}: N \tag{71}$$

We know what we want to do: to divide the parameter space into regions containing the same number of collisions. So we can define the boundary functions analogously to the 2 Particles and a wall system.

$$B_N(\varepsilon)\text{ is the solution for }\alpha\text{ of}\begin{cases} v_2^{(N)} = v_3^{(N)} & N\text{ odd} \\ v_1^{(N)} = v_2^{(N)} & N\text{ even}\end{cases}\quad. \tag{72}$$

Unfortunately, this is not as easy of a task as it was before. We were able to program Mathematica to solve for the boundary functions in the previous system, but it turned out to be a much more complicated task for the 3 particle system. *Mathematica* simply isn't good at solving for complicated combinations of variables such as α. Therefore, we seek another method to find the boundary functions for a given *N*.

C. Dynamical Systems and Fixed Points

We are motivated to formulate the boundary functions as a dynamical system from its success on 2 particles and a wall system. Let's follow the exact same procedure and define the new parameter α after *N* collisions

$$\begin{aligned} \alpha &\equiv \frac{v_3 - v_2}{v_1 - v_2} \\ a^{(N)} &\equiv \frac{v_3^{(N)} - v_2^{(N)}}{v_1^{(N)} - v_2^{(N)}} \end{aligned}\quad. \tag{73}$$

Now let's write the parameters after one and two collisions in terms of the initial parameter. Due to the more complicated form of α compared to *a*, it's a slightly difficult task, but still just simple algebra. It can be shown that the final parameters can be rewritten in terms of the initial parameter as

$$\begin{aligned} \alpha'(\alpha) &= \frac{1+\varepsilon}{2\varepsilon} - \frac{\alpha}{\varepsilon} \\ \alpha''(\alpha) &= \frac{2\varepsilon\left((1+\varepsilon) - 2\alpha\right)}{(\varepsilon-1)^2 - 2(1+\varepsilon)\alpha} \end{aligned}\quad. \tag{74}$$

Just like the 2 particles and a wall dynamical system, we can nest these functions together *N* times to find out how α evolves after *N* collisions.

$$\begin{aligned} F(\alpha) &\equiv \alpha'(\alpha) = \frac{1+\varepsilon}{2\varepsilon} - \frac{\alpha}{\varepsilon} \\ G(\alpha) &\equiv \alpha''(\alpha) = \frac{2\varepsilon\left((1+\varepsilon)-2\alpha\right)}{(\varepsilon-1)^2 - 2(1+\varepsilon)\alpha} \end{aligned} \quad . \tag{75}$$

As before, *F* takes the α parameter one collision into the future while *G* takes it two collisions into the future. It's clear once again that the parameter α after *N* collisions can be written as

$$\alpha^{(N)} = \begin{Bmatrix} F\left(G^{\frac{N-1}{2}}(\alpha)\right) & N \text{ odd} \\ G^{\frac{N}{2}}(\alpha) & N \text{ even} \end{Bmatrix} . \tag{76}$$

The Boundary functions can be rewritten for this new formulation as well by looking at the collision order. If *N* is odd then particle 1 and 2 have just collided. To find the boundary of the region having at most this many collisions it must be true that

$$v_2^{(N)} = v_3^{(N)} \Rightarrow \alpha^{(N)} = 0 . \tag{77}$$

If *N* is even then particles 2 and 3 must have just collided and it must be true that

$$v_1^{(N)} = v_2^{(N)} \Rightarrow \alpha^{(N)} = \pm\infty \Rightarrow \frac{1}{\alpha^{(N)}} = 0 . \tag{78}$$

By this logic we can thus rewrite the boundary functions in terms of $\alpha^{(N)}$.

$$B_N(\varepsilon) \text{ is the solution for } \alpha \text{ of } \begin{Bmatrix} \alpha^{(N)} = 0 & N \text{ odd} \\ \dfrac{1}{\alpha^{(N)}} = 0 & N \text{ even} \end{Bmatrix} . \tag{79}$$

Now the boundary functions are in a form such that *Mathematica* is able to easily compute them. Once again they are rational functions of epsilon.

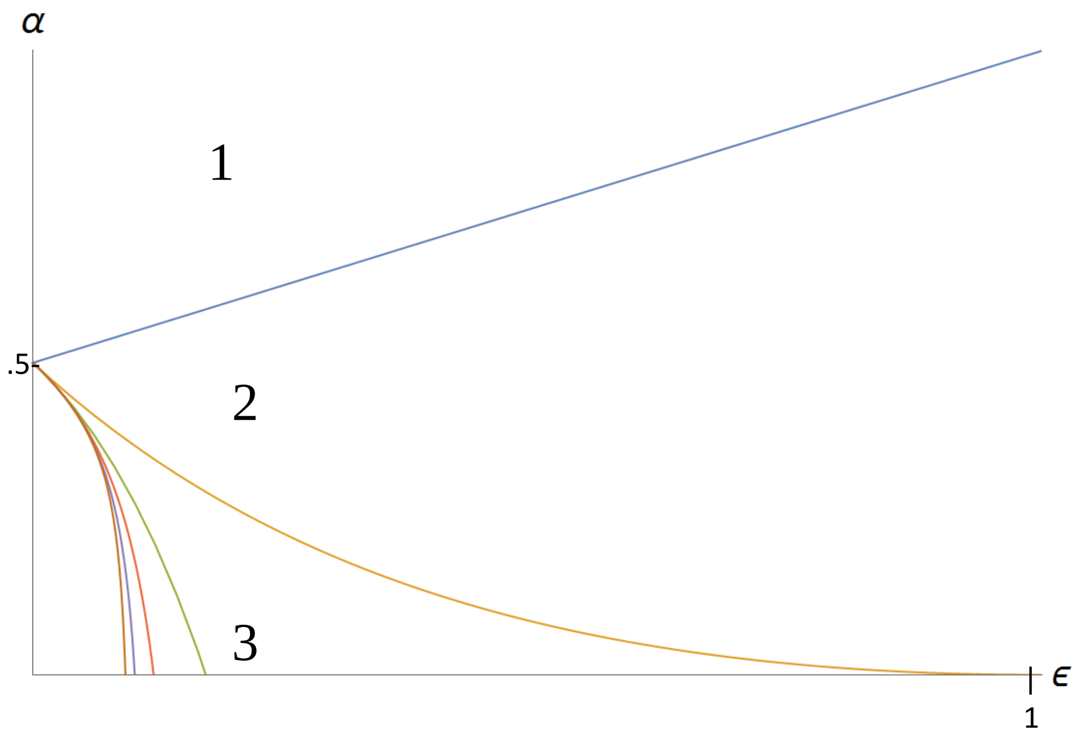

FIG.10. A plot of the first five boundary functions $B_1(\varepsilon) - B_5(\varepsilon)$. These boundary functions divide the α *vs.* ε parameter space into regions containing the indicated number of collisions.

$$
\begin{aligned}
B_1(\varepsilon) &= \frac{1+\varepsilon}{2} \\
B_2(\varepsilon) &= \frac{(-1+\varepsilon)^2}{2(1+\varepsilon)} \\
B_3(\varepsilon) &= \frac{1-5\varepsilon-5\varepsilon^2+\varepsilon^3}{2(-1+\varepsilon)^2} \\
B_4(\varepsilon) &= \frac{1-8\varepsilon-2\varepsilon^2-8\varepsilon^3+\varepsilon^4}{2(1-5\varepsilon-5\varepsilon^2+\varepsilon^3)}
\end{aligned} \tag{80}
$$

Now that any finite *N* boundary function can be computed using *Mathematica*, the next step is to find the inelastic collapse curve. Since we already have the 3 free particle system in terms of iterated functions, we can find the inelastic collapse curve by finding the fixed points of *G*, just like in 2 particles and a wall. Upon solving $G(\alpha)=\alpha$ for α and choosing the greater solution we obtain the inelastic collapse curve for 3 free particles.

$$\mathcal{B}(\varepsilon)=\frac{1}{4}(1+\varepsilon+\sqrt{1-14\varepsilon+\varepsilon^2})\,. \tag{81}$$

By requiring the inelastic Collapse Curve to have real values on its interior, we obtain the critical epsilon for 3 free particles by applying the quadratic formula to the radicand

$$\varepsilon<\varepsilon_c\equiv 7-4\sqrt{3}\approx .07\,. \tag{82}$$

The conditions for inelastic collapse for the 3 particle system is now obtained

$$\varepsilon<7-4\sqrt{3}\text{ and }\alpha<\mathcal{B}(\varepsilon)\Leftrightarrow\infty\text{ Collisions}\,. \tag{83}$$

The parameter space for 3 free particles is now complete! We can now celebrate by plotting a few finite boundary curves along with the inelastic collapse curve.

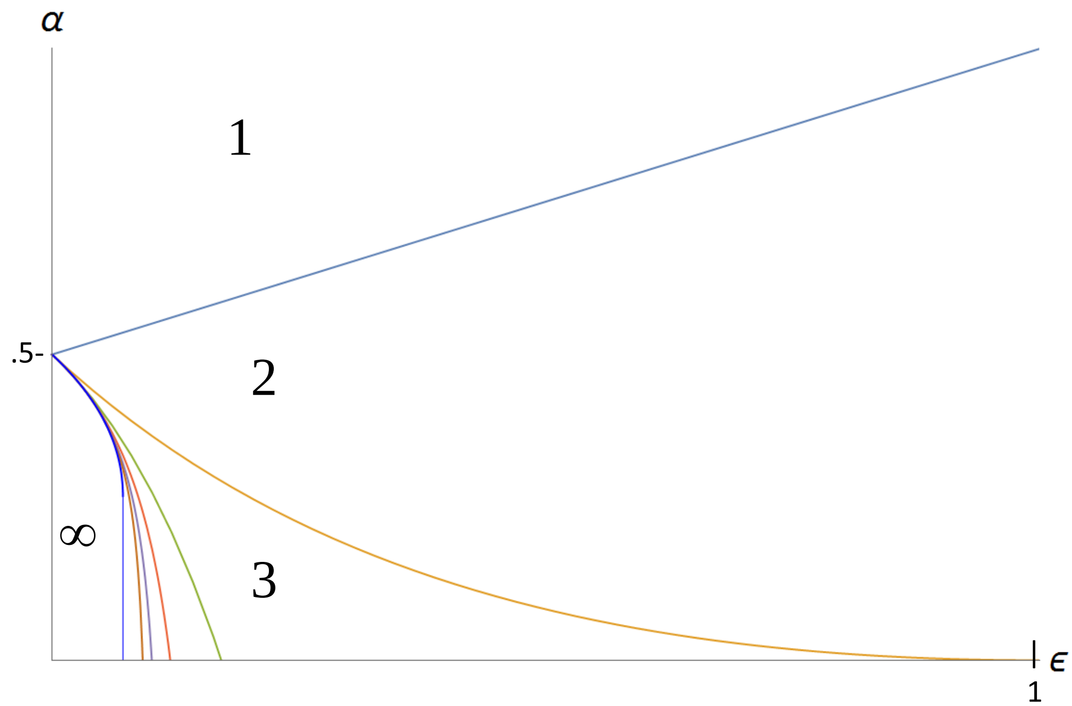

FIG.11. The completed parameter space of α $vs.\varepsilon$ for the 3 Free Particles system. The finite Boundary Functions can be drawn indefinitely approaching the Inelastic Collapse Curve. The region labeled ∞ is the Inelastic Collapse Region, the region where the system experiences infinite collisions.

D. Diagonalizing The Collision Matrix

The most direct, though most tedious method is simply to diagonalize the collision matrix right away, and compute both the finite and infinite boundary functions the same way. It can be shown using the same techniques in 2 Particles and a Wall that the boundary functions can be written as

$$B_N(\varepsilon)=\begin{cases}\dfrac{(g_2-1)\left(f_1(1+\varepsilon)+g_1(1-\varepsilon)-2\right)\lambda_2^{\frac{N-1}{2}}-(g_1-1)\left(f_2(1+\varepsilon)+g_2(1-\varepsilon)-2\right)\lambda_3^{\frac{N-1}{2}}}{(g_2-f_2)\left(f_1(1+\varepsilon)+g_1(1-\varepsilon)-2\right)\lambda_2^{\frac{N-1}{2}}-(g_1-f_1)(g_1-1)\left(f_2(1+\varepsilon)+g_2(1-\varepsilon)-2\right)\lambda_3^{\frac{N-1}{2}}} & N\text{ odd}\\[2ex] \dfrac{(g_1-f_1)(g_2-1)\lambda_2^{\frac{N}{2}}+(g_1-1)(f_2-g_2)\lambda_3^{\frac{N}{2}}}{(f_1-g_1)(f_2-g_2)(\lambda_2^{\frac{N}{2}}-\lambda_3^{\frac{N}{2}})} & N\text{ even}\end{cases} \quad (84)$$

where

$$\lambda_2=\frac{1}{8}\left(1-6\varepsilon+\varepsilon^2-(1+\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}\right)$$

$$\lambda_3=\frac{1}{8}\left(1-6\varepsilon+\varepsilon^2+(1+\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}\right)$$

$$f_1=-\frac{2\left(-1-\varepsilon+\sqrt{1-14\varepsilon+\varepsilon^2}\right)}{-1+6\varepsilon-\varepsilon^2+(1+\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}}$$

$$g_1=-\frac{1+8\varepsilon-\varepsilon^2-(1-\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}}{-1+6\varepsilon-\varepsilon^2+(1+\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}}$$

$$f_2=-\frac{2\left(1+\varepsilon+\sqrt{1-14\varepsilon+\varepsilon^2}\right)}{1-6\varepsilon+\varepsilon^2+(1+\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}}$$

$$g_2=-\frac{-1-8\varepsilon+\varepsilon^2-(1-\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}}{1-6\varepsilon+\varepsilon^2+(1+\varepsilon)\sqrt{1-14\varepsilon+\varepsilon^2}}.$$

Although they look VERY messy, they are actually rational functions of epsilon once again. We can compute both the finite and Inelastic Collapse curves to be

$$
\begin{aligned}
&B_1(\varepsilon) = \frac{1+\varepsilon}{2} \\
&B_2(\varepsilon) = \frac{(-1+\varepsilon)^2}{2(1+\varepsilon)} \\
&B_3(\varepsilon) = \frac{1-5\varepsilon-5\varepsilon^2+\varepsilon^3}{2(-1+\varepsilon)^2} \\
&B_4(\varepsilon) = \frac{1-8\varepsilon-2\varepsilon^2-8\varepsilon^3+\varepsilon^4}{2(1-5\varepsilon-5\varepsilon^2+\varepsilon^3)} \\
&\ldots \\
&\mathcal{B}(\varepsilon) = \frac{1}{4}(1+\varepsilon+\sqrt{1-14\varepsilon+\varepsilon^2}).
\end{aligned}
\tag{85}
$$

The Inelastic Collapse Curve was obtained by looking at the eigenvalues and simply taking the limit as *N*-> ∞ of the finite *N* boundary functions.

By requiring the inelastic Collapse Curve to have real values on its interior, we once again obtain the critical epsilon for 3 free particles by applying the quadratic formula to the radicand.

Through the same procedure in 2 particles in a wall, we can use the diagonalized collision matrix to write the finite boundary functions as rational functions of epsilon.

Because the collisions are ordered in the 3 Free Particle system as well, we are able to directly generalize our results from the 2 Particles and a Wall. We’ve found that the 3 particle system can experience an infinite number for all values of the coefficient of restitution less than some small critical value. We have not shown that these infinite collisions happen in a finite time.

VI. The Simulations

As a physicist, we should strive to verify our theoretical calculations with some experimental results. Unfortunately, we do not have any point particles colliding instantaneously with a constant coefficient of restitution. The next best thing is to write computer simulations to verify that the boundary functions really do divide the parameter spaces we've been studying into regions with the predicted number of collisions.

For the systems of 2 particles and a wall and 3 free particles, *Mathematica* simulations were written capable of telling us how many collisions we get when we input the initial parameters for each system. Both simulations use the velocity relations contained within the collision matrices and basic kinematic equations as their physics engines.

Fig. 11 and Fig. 12 show a sample output for both the 2 particles and a wall and 3 free particle systems respectively.

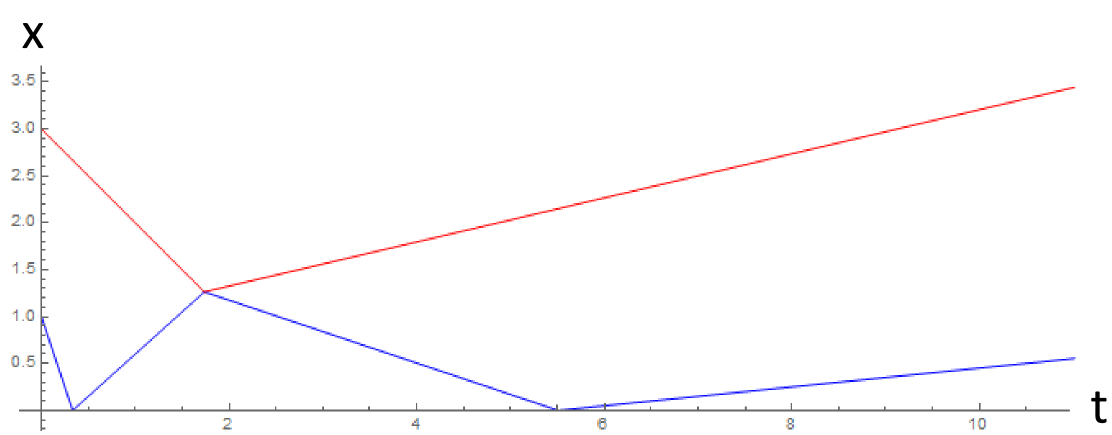

FIG.11. A sample output of the 2 particles and a wall simulation. The x vs. t graph represents 2 particles colliding with a fixed wall in one dimension. This example illustrates 3 collisions.

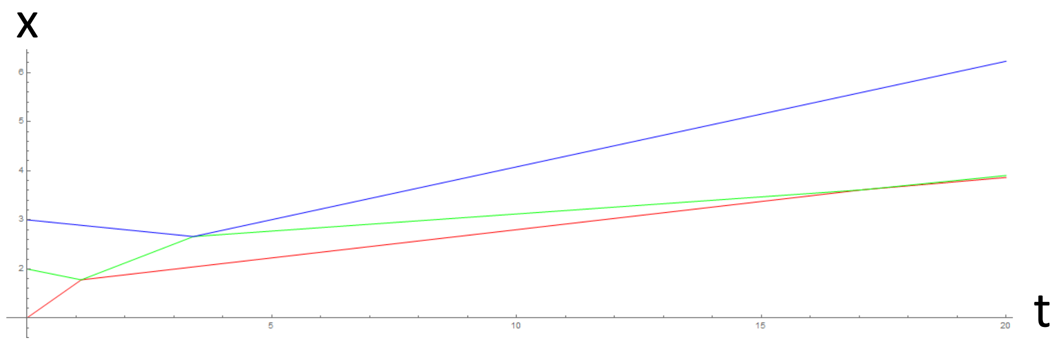

FIG.12. A sample output of the 3 free particles. The x vs. t graph represents 3 particles colliding in one dimension. This example illustrates 3 collisions.

Values of the initial velocities and the coefficient of restitution were chosen close to the boundary curves to see if they really did partition the parameter space. These values were input in to the simulations to determine the number of collisions for each initial configuration and instructed to stop if the number of collisions exceeded 10,000. Each point of the simulation output was color coded to represent a different number of collisions and overlaid the simulation data with the predicted boundary curves. Fig. 13 shows the result for the 2 particles and a wall system.

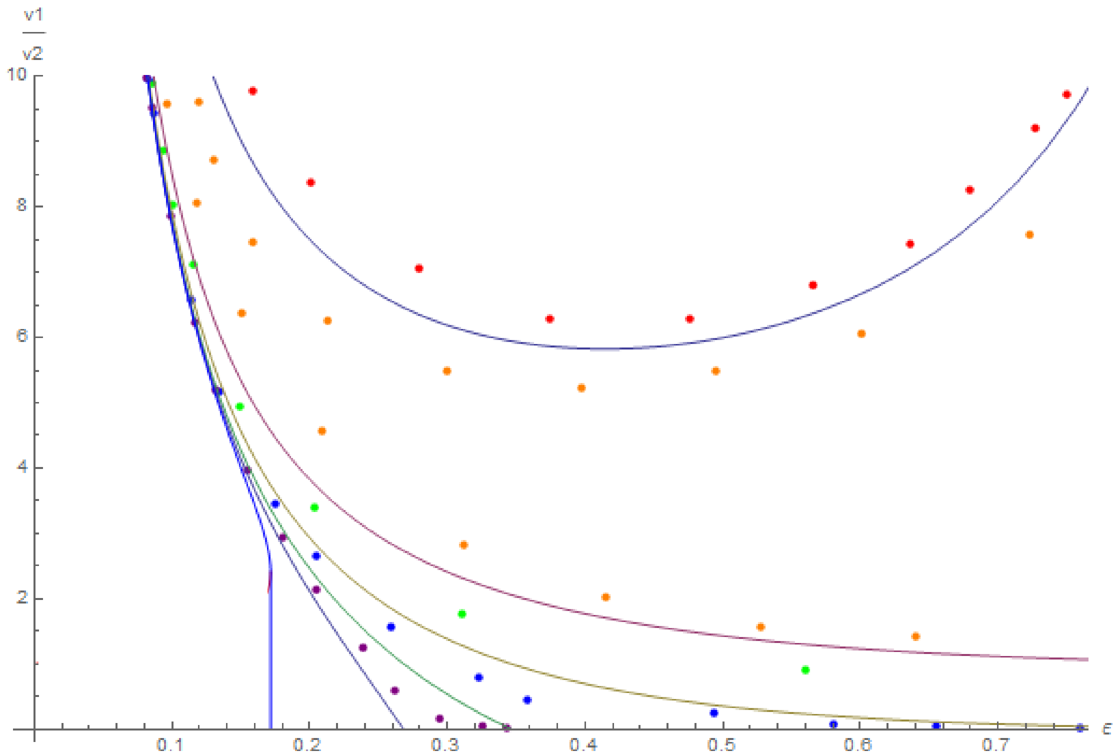

FIG.13. The theoretical parameter space for 2 particles and a wall overlaid with results from the simulation. As best as this data can show, the results of the simulation agree with the theoretical predictions. Red, Orange, Green, Blue, and Purple correspond to 2,3,4,5 and 6 collisions respectively.

We found that for every point tested, the simulations give the number of collisions indicated by the boundary functions. The $B_N(\epsilon)$ curves seem to partition the parameter space into regions with *N* collisions. We have run points inside the inelastic collapse region as well, and although we cannot run the simulation forever, we have ran it up to 10,000 collisions in this region by inserting a cutoff to tell the program to stop colliding particles after this *N* is reached. This is consistent with the blue $\mathcal{B}(\epsilon)$ curve being the inelastic collapse curve.

Although the results of the simulation supports our theoretical results, it cannot confirm it. We cannot, even in principle, run the simulation for all finite points and we certainly cannot run it infinitely long to test the infinite points. Nevertheless, these results are consistent with our analytical solutions and give us confidence in their validity.

VII. More than 3 Particles

The systems of 2 particles and a wall and 3 free particles have been solved analytically by exploiting the fact that the collisions are always ordered. In a system of 4 or more free particles in 1 dimension, this crucial fact is no longer true.

Consider a system of 4 free particles constructed by appending a 4^{th} particle next to particle 3. Let the first collision happen between particle 1 and 2 and even assume the second collision happens between particles 2 and 3. At this point we simply don't know which pair of particles will collide next without running a numerical simulation. Will particle 1 catch up to particle 2 again or will particle 3 crash into particle 4? Answering this question is complicated because it depends not only the initial velocities and the coefficient of restitution, but the initial positions as well. Therefore, no collision matrices can be employed with this system because we simply don't have the collision order.

The n>3 particle system cannot by investigated analytically in the manner we've previously employed. However, it was suspected that the higher n systems could still achieve inelastic collapse for a low enough coefficient of restitution. In fact, we though it should be easier for them. To simplify the n particle system, let the first particle's initial velocity be 1, all other particles be at rest. N vs. ε was plotted for a variety of initial velocities to see if there was evidence of inelastic collapse. Fig. 14 is a sample graph of N vs.ε from a 4 particle simulation showing that below some critical value of epsilon the number of collisions increases dramatically.

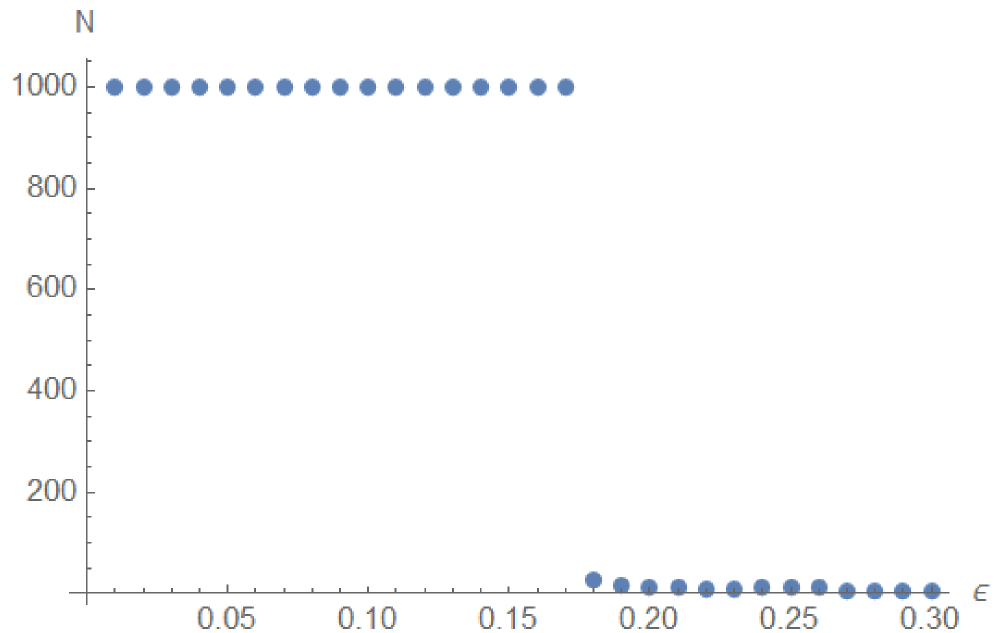

FIG.14. The output from a 4 free particle simulation illustrating that there appears to be a critical value of epsilon where inelastic collapse could occur.

However, it becomes difficult to partition the parameter space in these higher particle systems because it has been found through the simulations that the boundary functions are NOT monotonic for 4 particles! For a system of 3 particles the boundary functions are monotonic and decreasing epsilon *always* leads to increasing N as shown in Fig. 15.

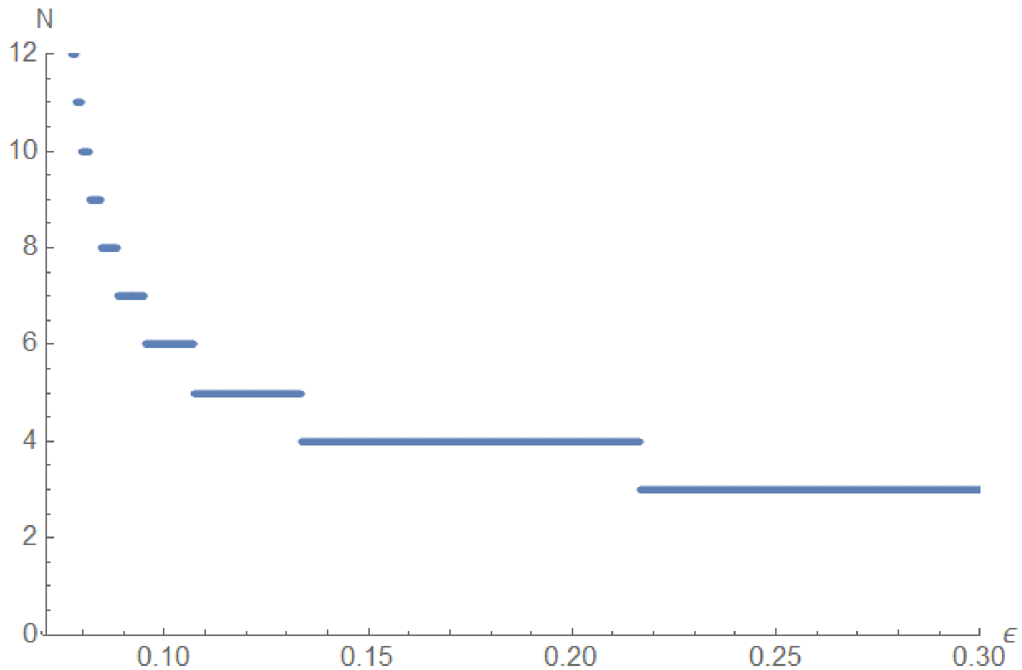

FIG.15. The output from a 3 free particle simulation illustrating that the number of collisions is a monotonic function of the coefficient of restitution.

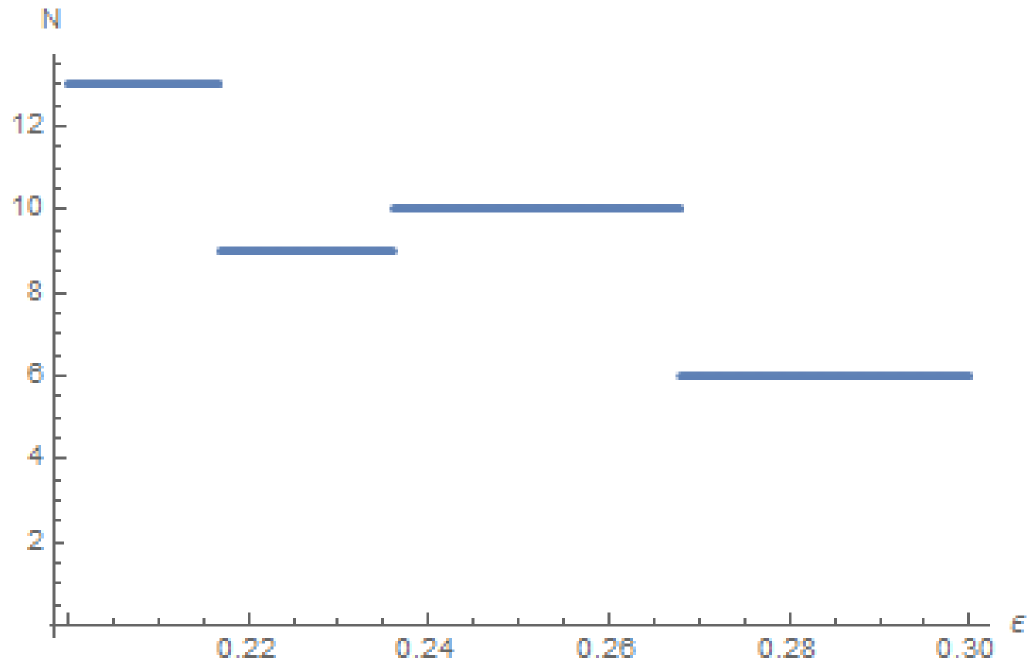

FIG.16. The output from a 4 free particle simulation illustrating that the number of collisions is NOT a monotonic function of the coefficient of restitution. A blow-up of Fig.14. “Bands” with the same collision number develop as a result of changing collision order.

Fig. 16 shows the number of collisions versus the coefficient of restitution for a three particle system with the middle particle initially at rest, a blow up of Fig. 14. The steps in the graph correspond to crossing the boundary functions. We see that decreasing epsilon *always* leads to an increasing number of collisions, consistent with the monotonicity of the boundary functions.

However, when the same types of graphs are made for 4 particles, “bands” of the same collision number appeared, not necessarily monotonic. The jumps in the number of collisions were found to represent changes in the collision order through simulational evidence. Each step represents a region of epsilon where the collision order is constant, and thus the gaps occur at values of epsilon where the collision order changes.

The non-monotonic graph of the *N* vs. ε graph in Fig. 16 suggests the boundary functions in the 4 particle parameter space intersect, and therefore analyzing them would be much more difficult.

If we only knew the collision order, then we could apply all our mathematical machinery to systems of more than three particles.

McNamara[2,6] and others have developed a new mathematical model for 4 or more particle systems called the Independent Collision Wave (ICW) model which takes a similar approach.

It has been shown by McNamara that there are critical values of the coefficient of restitution for which inelastic collapse will occur for general *n* free particle systems in 1 dimension.

VIII. Conclusion

We have studied the systems of 2 particles and a wall and 3 free particles in 1 dimension extensively. We obtained complete analytical solutions for these systems by defining parameter spaces of the coefficient of restitution and initial velocities and breaking them up into regions that result in the same number of collisions. We found the boundaries between these regions by applying stopping conditions to the velocities and deriving analytic forms of the Boundary Functions. These finite *N* boundary functions were found to converge to the Inelastic Collapse curve for which the coefficient of restitution lower than the critical values of $3-2\sqrt{2}$ for 2 particles and a wall and $7-4\sqrt{3}$ for 3 free particles led to the particles colliding infinitely many times. We verified these results by *Mathematica* simulation up to 10,000 collisions.

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