Perfect Polynomials

modulo 2

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Outline

1. Definitions
   - modulo 2
   - What is "Perfect"?
   - Perfect Polynomials

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   - What is "Perfect"?
   - Perfect Polynomials
2. Previous Research
   - Others'
   - Ours

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- In simple terms, 'a mod b' gives the remainder when integer a is divided by non-zero integer b.

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- $13 \equiv 1$ and $54678 \equiv 0 \pmod 2$

Perfect Numbers

Sigma function σ

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Lower case Greek letter sigma (σ) symbolizes an arithmetic function that sums the positive divisors of a positive integer.

$$\sigma(n) = \sum_{d|n} d$$

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- Hence, 6 is perfect.

Continuing on σ

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σ is multiplicative over integers.
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Polynomials mod 2

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- $x^2 + 3x + 2 \equiv x^2 + x \pmod 2$
- So $\sigma(x^2 + x) = x^2 + x \pmod 2$
- $x^2 + x$ is a perfect polynomial mod 2.

Canaday, Gallardo and Rahavandrainy

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- **Gallardo and Rahavandrainy**
- Proved that odd perfects have at least 5 distinct irreducible factors.

Degree	Factorization into Irreducibles
5	$T(T+1)^2(T^2+T+1)$
	$T^2(T+1)(T^2+T+1)$
11	$T(T+1)^2(T^2+T+1)^2(T^4+T+1)$
	$T^2(T+1)(T^2+T+1)^2(T^4+T+1)$
	$T^3(T+1)^4(T^4+T^3+1)$
	$T^4(T+1)^3(T^4+T^3+T^2+T+1)$
15	$T^3(T+1)^6(T^3+T+1)(T^3+T^2+1)$
	$T^6(T+1)^3(T^3+T+1)(T^3+T^2+1)$
16	$T^4(T+1)^4(T^4+T^3+1)(T^4+T^3+T^2+T+1)$
20	$T^4(T+1)^6(T^3+T+1)(T^3+T^2+1)(T^4+T^3+T^2+T+1)$
	$T^6(T+1)^4(T^3+T+1)(T^3+T^2+1)(T^4+T^3+1)$

Figure: Canaday's list for perfects

The Algorithm to Find the Perfect Polynomials

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The Algorithm to Find the Perfect Polynomials

- Check if σB = B. Output B.
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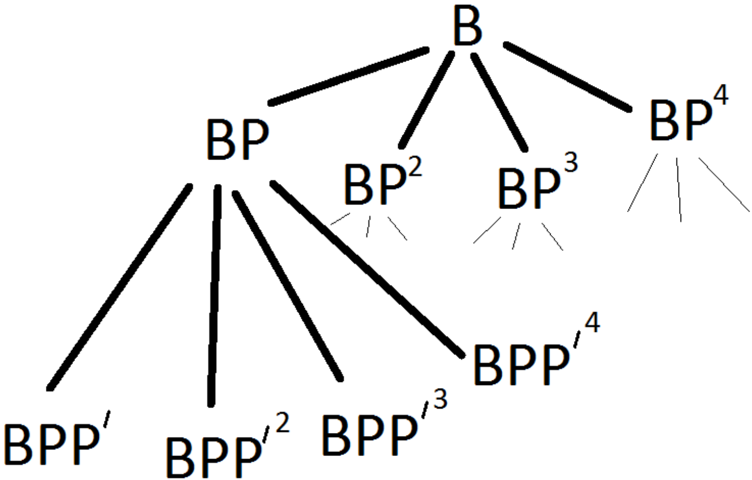
- If gcd(B, D) $>$ 1, then stop. No output!
- If the polynomial passes the test on step 3, then let P be the greatest factor of D.
- Restart the algorithm taking BP, BP^2, BP^3, ..., BP^k where degree of BP^k $< K$.


B
BP
BP2
BP3
BP4
BPP'
BPP'2
BPP'3
BPP'4

Beginning steps - primPerf()

```
def primPerf(B):
    if B == sumDivs4(B):
        return B
    else:
        D = (sumDivs4(B)/gcd(B, sumDivs4(B)))
        if gcd(D,B) != 1:
            return False
        else:
            F = D.factor()
            P = F[len(F)-1][0]
            check = False
            K = 1
            while (B*(P^K)).degree() <= 1000:
                check = primPerf(B*(P^K))
                if check == False:
                    K = K + 1
                else:
                    return primPerf((B*(P^K)))
                    break
```

Results up to degree 200

$x \times (x+1)^2 \times (x^2+x+1)$
$x \times (x+1)^2 \times (x^2+x+1)^2 \times (x^4+x+1)$
$(x+1) \times x^2 \times (x^2+x+1)$
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$(x+1)^4 \times x^6 \times (x^3+x+1) \times (x^3+x^2+1) \times (x^4+x^3+1)$
$x \times (x+1)$
$x^3 \times (x+1)^3$
$x^7 \times (x+1)^7$
$x^{15} \times (x+1)^{15}$
$x^{31} \times (x+1)^{31}$ and $x^{63} \times (x+1)^{63}$

Speed

```
def sigma1(x, y):
  return (x^(y+1) - 1)/(x - 1)

def sigma2(x, y):
   sum = 0
   for pow in range(0, y+1):
     sum = sum + (x^pow)
  return sum
```

*sigma*1 and *sigma*2 speed testing

Dynamic Programming

```
import time
tic = time.clock()
sum = x^30
found = primPerf(sum)
if type(found)== type(x):
    print found, "=", found.factor()
toc = time.clock()
toc - tic

  0.000747000000182608

perfFinder(15)

  1 = 1
  degree =  0

  x^2 + x = x * (x + 1)
  degree =  2

  x^2 + x = x * (x + 1)
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```

Figure: FAST!

Summary

- A perfect polynomial equals the sum of its divisors.
- As Canaday thought there are no odd perfect polynomials up to degree 200.
- My program is relatively fast and finds the perfect polynomials.

- Future Plans
  - To check higher degrees
  - Show odd perfect polynomials mod 2 have at least 6 factors
  - Work on a paper

For Further Information

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Duke Mathematical Journal, 8(4):721–737, 1941

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There is no odd perfect polynomial over F_2 with four prime factors
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Thank You!

(Any Questions?)